Estimation of Dynamic Discrete Choice Models in Continuous Time

Peter Arcidiacono[1] Patrick Bayer[1] Jason R. Blevins[2]
Paul B. Ellickson[3]

[1]Duke University and NBER

[2]Duke University

[3]Simon Graduate School of Business, University of Rochester

November 11, 2009

MOTIVATION

- Dynamic discrete choice models have a long tradition.
- Estimation of dynamic games is much more recent.
  - Full-solution (NFXP) methods seemed infeasible.
  - Two-step (CCP) approaches proved useful.
- We can now *estimate* complex games but *solving* them remains difficult.
  - One benefit of structural work is the ability to perform counterfactual experiments.
  - Often, researchers are forced to use a much simpler model or forgo them altogether.
- We propose a continuous-time model which:
  - is as empirically tractable as discrete-time counterparts,
  - can be solved quickly even with many players,
  - can be estimated with continuous or discrete time data, and
  - can be estimated using two-step methods.

Computation of Dynamic Discrete Games

- Classic example: framework of Ericson and Pakes (1995) and Pakes and McGuire (1994).
  - Designed to track an oligopolistic industry over time.
  - Firm heterogeneity modeled as a firm-specific state (quality).
  - Incumbent firms decide whether to remain in the industry and how much to invest.
  - Potential entrants decide whether or not to enter.
  - Active firms compete in a product market.
- Hard to handle more than a few firms and a single state variable.
- There is a curse of dimensionality in calculating firms' expectations over future states.
  - If N players can each move to one of K states, we must consider K^N possible outcomes.
  - Due to the simultaneity of moves.
  - Sharply limits scalability (size and heterogeneity).

Methods for Reducing Computation

1. Pakes and McGuire (2001)
    - Stochastic algorithm.
    - Approximation.
    - Targets recurrent class of states.
2. Doraszelski and Judd (2008)
    - Continuous time.
    - Eliminates simultaneous moves.
3. Weintraub, Benkard, and Van Roy (2008)
    - Alternative equilibrium concept: oblivious equilibrium.
    - Approximates Markov perfect equilibrium.

Continuous Time

- We follow Doraszelski and Judd (2008) in using continuous time.
- Only a single player's state may change in any given instant.
    - Result: sum over $(K-1) \times N$ states instead of K^N.
    - Doraszelski and Judd (2008) show that for $N = 14$, computation is 30,000 times faster in continuous time.
    - Solving for equilibrium: 20 minutes vs. 1 year.
- Consider a modified model for empirical purposes:
    - Agents make discrete choices at random times in a random order.
    - Move opportunities arrive according to a collection of competing Poisson processes.
    - Simultaneous moves are possible, but such events have measure zero.

Practical Concerns

- Move arrivals in continuous time.
  - Closest to discrete time when λ is fixed (e.g., $\lambda \equiv 1$).
  - We estimate λ.
- Players in our model make *discrete* choices.
  - Entry, exit, and discrete investments, such as building a plant.
  - Not well-suited for R&D without modification.
- Most data sets are sampled at regular intervals.
  - Our continuous-time model can be aggregated to the observed frequency for estimation.
  - We can also account for passive "non-moves".

OVERVIEW

- Notation and definitions.
- Single-agent dynamic discrete choice models.
    - Rust (1987) replacement investment problem.
    - Passive moves and time aggregation.
    - CCP estimation.
    - Monte Carlo experiments.
- Dynamic discrete games.
    - Pakes and McGuire (1994) quality ladder model.
    - Monte Carlo experiments.
- Extensions and future work.

Markov Jump Processes

- A Markov jump process is a stochastic process X_t indexed by $t \in [0, \infty)$.
- X_t takes values on a discrete state space $\mathcal{X}$.
- Sample paths are piecewise-constant right-continuous functions.
- Jump intervals are independent and exponentially distributed.
- The *intensity matrix* $Q = (q_{ij})$ determines the evolution of the process:
  - q_{ii} is the rate at which X_t leaves i.
  - q_{ij} is the rate at which X_t moves from i to j.
- Q satisfies the following:
  - $q_{ij} \geq 0$ for all $i \neq j$,
  - $q_{ii} = -\sum_{j \neq i} q_{ij}$ for all i,
  - $\sum_j q_{ij} = 0$ for all i.

INTENSITY MATRIX

- A representative intensity matrix:

$$Q = \begin{bmatrix} -q_{11} & q_{12} & \cdots & q_{1K} \\ q_{21} & -q_{22} & \cdots & q_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ q_{K1} & q_{K2} & \cdots & -q_{KK}. \end{bmatrix}$$

- Transitions out of state i occur as Exponential(q_{ii}).
- Conditional on jumping from i, the state moves to j with probability proportional to q_{ij}:

$$P_{ij} = \frac{q_{ij}}{\sum_{k \neq i} q_{ik}}.$$

Transition Matrix

- We also make use of the transition matrix $P(t)$.
- Let $P_{ij}(t) = \Pr(X_{t+s} = j \mid X_s = i)$ and $P(t) = (P_{ij}(t))$.
- $P(t)$ is then the unique solution to the forward equations:

$$P'(t) = P(t)Q, \quad P(0) = I$$

which can be written in terms of the matrix exponential

$$P(t) = \mathrm{e}^{tQ} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!}.$$

- There are many known algorithms for computing $P(t)$.
- Note: the $1 \times K$ limiting (ergodic) distribution μ^∞ solves the linear system $\mu^\infty Q = 0$.

Competing Processes

- Consider two competing jump processes on $\mathcal{X}$.
- Two intensity matrices Q_0 and Q_1.
- The combined process has intensity matrix $Q = Q_0 + Q_1$.
- When the current state is k, the process leaves at rate $q_{0kk} + q_{1kk}$.
- Upon leaving, the state jumps from k to l at rate $q_{0kl} + q_{1kl}$.
- We build dynamic discrete choice models and dynamic games by combining multiple jump processes:
    - Q_0 encompasses exogenous state changes,
    - Q_i encompasses state changes due to the actions of agent i.

SINGLE-AGENT DYNAMIC DISCRETE CHOICE

- Consider a single agent controlling the Markov jump process X_t, $t \geq 0$.
- The agent is forward-looking with discount rate ρ.
- Two competing jump processes drive the model:
  1. A continuous-time Markov jump process on $\mathcal{X} = \{1, \ldots, K\}$ with intensity matrix Q_0 driving exogenous state changes.
  2. A Poisson process with rate λ that governs when agent can move.
- When an opportunity to move arises, conditional on the current state, the agent chooses an action a from the choice set $\mathcal{A} = \{1, \ldots, J\}$.

PAYOFFS & MOVES

- The agent receives flow utility u_k in state k.
- If the model remains in state k over $[0, \tau)$, the PDV (from $t = 0$) is

$$\int_0^\tau e^{-\rho t} u_k \, dt.$$

- If a move arrives in state k and the agent chooses action $j \in \mathcal{A}$, the instantaneous payoff is $\psi_{jk} + \varepsilon_j$, where
  - ψ_{jk} is the mean instantaneous payoff/cost, and
  - ε_j is a choice-specific shock (*iid* TIEV).
- Let σ_{jk} denote the probability the agent chooses j in state k, and let w_{jk} be the continuation value associated with that choice.

VALUE FUNCTION

- We can write the value function recursively as

$$v_k = \mathrm{E}\left[\int_0^\tau \mathrm{e}^{-\rho t}\, u_k\, dt + \mathrm{e}^{-\rho\tau}\, \frac{1}{\lambda + q_{kk}} \left(\sum_{l \neq k} q_{kl} v_l + \lambda \max_j \left\{\psi_{jk} + \varepsilon_j + w_{jk}\right\}\right)\right]$$

  where the expectation is with respect to τ and ε.

- The optimal policy rule $\delta : \mathcal{X} \times \mathcal{E} \to \mathcal{A}$ satisfies

$$\delta(k, \varepsilon) = j \iff \psi_{jk} + \varepsilon_j + w_{jk} \geq \psi_{lk} + \varepsilon_l + w_{lk} \quad \forall l \in \mathcal{A}.$$

- The conditional choice probabilities (CCPs) σ_{jk} are

$$\sigma_{jk} = \Pr\left(\delta(k, \varepsilon) = j \mid k\right).$$

Rust (1987) Example

- The state space represents accumulated mileage: $\mathcal{X} = \{1, \dots, K\}$.
- State transitions: the $K \times K$ intensity matrix on $\mathcal{X}$ is

$$Q_0 = \begin{bmatrix} -q_1 - q_2 & q_1 & q_2 & 0 & \dots & 0 \\ 0 & -q_1 - q_2 & q_1 & q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -q_1 - q_2 & q_1 & q_2 \\ 0 & 0 & \dots & 0 & -q_1 - q_2 & q_1 + q_2 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

- At stochastic points in time, arriving at some rate λ, Harold Zurcher decides whether or not to replace the engine: $\mathcal{A} = \{0, 1\}$.
- This is a cost minimization problem:
  - flow utility $u_k = -\beta k$ received upon continuation ($j = 0$),
  - replacement cost c paid upon replacement ($j = 1$),
  - plus iid shocks in each case.

Value Function

- Instantaneous payoffs:

$$\psi_{jk} = \begin{cases} 0 & \text{if } j = 0, \\ -c & \text{if } j = 1. \end{cases}$$

- Continuation values:

$$w_{jk} = \begin{cases} v_k & \text{if } j = 0, \\ v_1 & \text{if } j = 1. \end{cases}$$

- Value function:

$$v_k = \mathrm{E}\left[\int_0^\tau \mathrm{e}^{-\rho t}\, u_k\, dt + \mathrm{e}^{-\rho\tau}\left(\frac{q_1}{\lambda + q_1 + q_2} v_{k+1} + \frac{q_2}{\lambda + q_1 + q_2} v_{k+2} \right.\right.$$
$$\left.\left. \frac{\lambda}{\lambda + q_1 + q_2} \max\left\{\varepsilon_0 + v_k, -c + \varepsilon_1 + v_1\right\}\right)\right].$$

SIMPLIFICATIONS

- Assuming $\varepsilon_j \sim$ TIEV:

$$v_k = \mathrm{E}\left[\int_0^\tau e^{-\rho t}\, u_k\, dt + e^{-\rho\tau}\left(\frac{q_1}{\lambda + q_1 + q_2} v_{k+1} + \frac{q_2}{\lambda + q_1 + q_2} v_{k+2} \right.\right.$$
$$\left.\left. \frac{\lambda}{\lambda + q_1 + q_2} \ln\left(\exp(v_k) + \exp(v_1 - c)\right)\right)\right].$$

- Conditional choice probabilities:

$$\sigma_{jk} = \begin{cases} \frac{\exp(v_k - v_1 + c)}{\exp(v_k - v_1 + c) + 1} & \text{if } j = 0, \\ \frac{1}{\exp(v_k - v_1 + c) + 1} & \text{if } j = 1. \end{cases}$$

ESTIMATION

Consider a sample of T tuples $(\tau_t, i_t, a_t, x_t, x_t')$, each describing a jump or move where, for each observation t:

- τ_t is the holding time since the previous event,
- i_t is the player index associated with this event ($i_t = 0$ indicates a move by nature),
- a_t is the action taken by player i_t,
- x_t denotes the state at the time of the event, and
- x_t' denotes the state immediately after the event.

LIKELIHOOD

- Let $\ell_t(\theta)$ denote the likelihood of observation t given θ.
- Let $g(\tau; \lambda)$ and $G(\tau; \lambda)$ denote the pdf and cdf of Exponential(λ).
- For two states x and x', let
  - $q(x, x'; \theta)$ denote the corresponding element of the intensity matrix, and
  - $p(x, x'; \theta)$ denote the corresponding element of the jump matrix P.
- Finally, let $\sigma(i_t, a_t, x_t; \theta)$ denote the probability of player i_t choosing a_t in state x_t.
- $\ell_t(\theta)$ will then differ for moves and jumps.

Likelihood for Jumps and Moves

- For moves, indicated by $i_t > 0$, the likelihood is

$$\ell_t(\theta) = g(\tau_t; \lambda) \cdot \sigma(i_t, a_t, x_t; \theta) \cdot [1 - G(\tau_t; q(x_t, x_t; \theta))] .$$

- For jumps, indicated by $i_t = 0$, the likelihood is

$$\ell_t(\theta) = g(\tau_t; q(x_t, x_t; \theta)) \cdot p(x_t, x_t'; \theta) \cdot [1 - G(\tau_t; \lambda)] .$$

- The log-likelihood of the sample is

$$\begin{aligned} \ln L_T(\theta) = & \sum_{t=1}^{T} 1\{i_t > 0\} \left[\ln g(\tau_t; \lambda) + \ln \sigma(i_t, a_t, x_t; \theta) + \ln [1 - G(\tau_t; q(x_t, x_t; \theta))]\right] \\ & + \sum_{t=1}^{T} 1\{i_t = 0\} \left[\ln g(\tau_t; q(x_t, x_t; \theta)) + \ln p(x_t, x_t'; \theta) + \ln [1 - G(\tau_t; N\lambda)]\right] . \end{aligned}$$

- The model can be then estimated either by full solution or via a two-step (CCP) approach.

EXTENSIONS

- Passive moves
  - Suppose we don't observe actions for which $a = 0$.
  - We might not see Harold "decide not" to replace the engine.
  - We see a truncated joint distribution of arrival times and actions.
  - We can adjust for truncation using structure implied by model.
- Time aggregation
  - Suppose observations are sampled at regular intervals of length Δ.
  - We cannot see the actual sequence of moves or move times.
  - Estimate using the transition matrix $P(\Delta)$.
  - Model predictions are aggregated to observed frequency.

Monte Carlo Experiments: NFXP Estimation

Sampling	n	q_1	q_2	λ	β	c
Population	∞	0.150	0.050	0.200	1.000	1.250
Continuous Time	10,000	0.150	0.050	0.200	1.008	1.254
		(0.002)	(0.001)	(0.003)	(0.068)	(0.053)
Passive Moves	7,176	0.150	0.050	0.204	1.010	1.271
		(0.002)	(0.001)	(0.020)	(0.127)	(0.126)
$\Delta = 0.625$	40,000	0.150	0.050	0.206	1.060	1.324
		(0.003)	(0.002)	(0.031)	(0.270)	(0.253)
$\Delta = 1.25$	20,000	0.150	0.050	0.203	1.061	1.301
		(0.003)	(0.002)	(0.028)	(0.228)	(0.225)
$\Delta = 2.5$	10,000	0.150	0.050	0.204	1.008	1.224
		(0.004)	(0.002)	(0.031)	(0.321)	(0.381)
$\Delta = 5.0$	5,000	0.150	0.050	0.203	1.084	1.311
		(0.007)	(0.003)	(0.035)	(0.315)	(0.393)
$\Delta = 10.0$	2,500	0.157	0.048	0.204	0.996	1.138
		(0.018)	(0.007)	(0.028)	(0.398)	(0.601)

Table: Single player Monte Carlo results: NFXP estimation ($T = 25,000$).

Monte Carlo Experiments: CCP Estimation

Sampling	n	q_1	q_2	λ	β	c
Population	∞	0.150	0.050	0.200	1.000	1.250
Continuous Time	10,000	0.150	0.050	0.200	1.015	1.256
		(0.002)	(0.001)	(0.003)	(0.064)	(0.053)
Passive Moves	7,176	0.150	0.050	0.187	0.829	1.157
		(0.002)	(0.001)	(0.012)	(0.148)	(0.094)
$\Delta = 0.625$	40,000	0.150	0.050	0.215	1.097	1.339
		(0.003)	(0.002)	(0.092)	(0.310)	(0.351)
$\Delta = 1.25$	20,000	0.150	0.050	0.210	1.079	1.332
		(0.003)	(0.002)	(0.053)	(0.290)	(0.338)
$\Delta = 2.5$	10,000	0.150	0.050	0.235	1.099	1.347
		(0.004)	(0.002)	(0.218)	(0.342)	(0.460)
$\Delta = 5.0$	5,000	0.207	0.065	0.282	1.226	1.303
		(0.347)	(0.093)	(0.387)	(0.759)	(0.515)
$\Delta = 10.0$	2,500	0.171	0.051	0.288	1.195	1.421
		(0.172)	(0.016)	(0.347)	(0.666)	(0.765)

Table: Single player Monte Carlo results: CCP estimation ($T = 25,000$).

Dynamic Discrete Games

- Consider a game with N players, $i = 1, \ldots, N$
- The main quantities of interest are
  - the flow utility in state k: u_{ik}
  - the choice probabilities: σ_{ijk}, and
  - the instantaneous payoffs: $\psi_{ijk} + \varepsilon_{ijk}$.
- Moves by nature are captured by a single process on $\mathcal{X}$ with intensity matrix Q_0.
- There are now N competing processes with rates λ_i generating move opportunities for each player.
- Beliefs about rivals' actions become part of the aggregate intensity matrix:

$$Q_{-i} \equiv Q_0 + \sum_{j \neq i} Q_j$$

- We use the next-*move* representation of the value function.

Value Function, Policies & Equilibrium

- The value function for player i is

$$v_{ik} = \mathrm{E}\left[\int_0^{\tau_i} \mathrm{e}^{-\rho_i t} \sum_{l=1}^{K} P_{kl}^{-i}(t) u_{il}\, dt + \mathrm{e}^{-\rho \tau_i} \sum_{l=1}^{K} P_{kl}^{-i}(\tau_i) \max_j \left\{ \psi_{ijl} + \varepsilon_{ij} + w_{ijl} \right\}\right],$$

  where $P^{-i}(t)$ is constructed using the intensity matrix Q_{-i}.

- The optimal policy rule for player i satisfies

$$\delta_i(k, \varepsilon_i) = j \iff \psi_{ijk} + \varepsilon_{ij} + w_{ijk} \geq \psi_{ilk} + \varepsilon_{il} + w_{ilk} \quad \forall l \in \mathcal{A}_i, \tag{1}$$

  while their CCPs satisfy

$$\sigma_{ijk} = \Pr\left(\delta_i(k, \varepsilon_i) = j \mid k\right). \tag{2}$$

- A *Markov perfect equilibrium* is a collection of policy rules $\{\delta_i\}$ and beliefs $\{\sigma_{ijk}\}$ such that both (1) and (2) hold for all i.

Quality Ladder Example

- Consider the classic quality ladder model of Ericson and Pakes (1995) and Pakes and McGuire (1994).
- N players compete in a single differentiated products market.
  - Firm i provides quality $\omega_i \in \Omega$, where $\Omega = \{1, 2, \dots, \bar{\omega}, \bar{\omega} + 1\}$.
  - $\bar{\omega} + 1$ denotes the (inactive) potential entrant state.
- The value of the outside alternative increases at rate γ (nature).
- Product market competition is a standard discrete choice logit model.
  - Continuum of consumers with measure $M > 0$.
  - Utility for product i for consumer j is $g(\omega_i) - p_i + \varepsilon_{ij}$, where the g function keeps profits bounded and firms on the grid.
  - Competition is Nash in prices.
- Market shares follow the standard logit formula and profits can be pre-calculated by solving the relevant first-order conditions.

Incumbent Firms

- Incumbent firms have three choices when moving:
  1. continue without investing at no cost,
  2. invest κ to increase quality from ω_i to $\omega_i' = \max\{\omega_i + 1, \bar{\omega}\}$, or
  3. exit and receive scrap payment η.
- Each cost includes a private shock $\varepsilon_{ijk} \sim$ TIEV.
- The instantaneous payoffs ψ_{ijk} and continuation values are:

$$\psi_{ijk} = \begin{cases} 0 & \text{if } j = 0, \\ -\kappa & \text{if } j = 1, \\ \eta & \text{if } j = 2, \end{cases} \quad w_{ijk} = \begin{cases} v_{ijk} & \text{if } j = 0, \\ v_{ijk'} & \text{if } j = 1, \\ 0 & \text{if } j = 2. \end{cases}$$

Incumbent Firms

- The value function for an incumbent in state k is

$$v_{ik} = E\left[\int_0^{\tau_i} \mathrm{e}^{-\rho t} \sum_{l=1}^{K} P_{kl}^{-i}(t)\pi_{il}\,dt \right.$$
$$\left. + \mathrm{e}^{-\rho\tau_i} \sum_{l=1}^{K} P_{kl}^{-i}(\tau_i) \max\left\{v_{ik} + \varepsilon_{i0}, -\kappa + v_{ik'} + \varepsilon_{i1}, \eta + \varepsilon_{i2}\right\}\right]$$

- Conditional upon moving in state k, incumbents face

$$\max\left\{v_{ik} + \varepsilon_{i0}, -\kappa + v_{ik'} + \varepsilon_{i1}, \eta + \varepsilon_{i2}\right\}$$

resulting in closed form CCPs

$$\sigma_{i0k} = \frac{\exp(v_{ik})}{\exp(v_{ik}) + \exp(-\kappa + v_{ik'}) + \exp(\eta)}$$
$$\sigma_{i1k} = \frac{\exp(-\kappa + v_{ik'})}{\exp(v_{ik}) + \exp(-\kappa + v_{ik'}) + \exp(\eta)}$$
$$\sigma_{i2k} = 1 - \sigma_{i0k} - \sigma_{i1k}$$

POTENTIAL ENTRANTS

- If the number of active firms is $n < N$, a potential entrant gets the opportunity to enter at rate λ
- A potential entrant i has two choices
  1. enter ($a_i = 1$) at state $\omega^e \in \Omega$ and pay entry cost η^e, or
  2. choose not to enter ($a_i = 0$), at no cost.
- Both costs include iid shocks $\varepsilon^e_{ijk} \sim$ TIEV.

Potential Entrants

- Instantaneous payoffs and continuation values are

$$\psi_{ijk} = \begin{cases} 0 & \text{if } j = 0, \\ -\eta^e & \text{if } j = 1, \end{cases} \quad \text{and} \quad w_{ijk} = \begin{cases} 0 & \text{if } j = 0, \\ v_{ik'} & \text{if } j = 1, \end{cases}$$

  where k' is the resulting market state after entry.
- Potential entrants choose actions according to

$$\max\{\varepsilon^e_{i0}, -\eta^e + v_{ik'} + \varepsilon^e_{i1}\}$$

  yielding entry CCPs

$$\sigma_{i1k} = \frac{\exp(v_{ik'} - \eta^e)}{1 + \exp(v_{ik'} - \eta^e)}$$
$$\sigma_{i0k} = \frac{1}{1 + \exp(v_{ik'} - \eta^e)}$$

Monte Carlo Experiments: NFXP Estimation

N	K	M	n_{avg}	λ	γ	κ	η	η^e
Population				1.800	0.200	0.800	4.000	5.000
10	80,080	5.0	6.62	1.820	0.201	0.798	3.986	4.967
				(0.005)	(0.001)	(0.026)	(0.204)	(0.171)
12	222,768	8.0	8.29	1.821	0.201	0.798	4.010	5.007
				(0.006)	(0.001)	(0.024)	(0.192)	(0.163)
14	542,640	10.0	9.32	1.821	0.200	0.801	4.043	5.044
				(0.006)	(0.001)	(0.031)	(0.184)	(0.157)
16	1,193,808	13.0	10.89	1.822	0.200	0.837	3.967	5.152
				(0.005)	(0.001)	(0.103)	(0.223)	(0.402)
18	2,422,728	17.0	12.90	1.821	0.200	0.824	3.986	5.081
				(0.007)	(0.002)	(0.061)	(0.162)	(0.273)
20	4,604,600	21.0	14.92	1.817	0.201	0.779	4.051	5.166
				(0.006)	(0.003)	(0.025)	(0.174)	(0.186)

Table: Quality ladder Monte Carlo results: NFXP estimation.

Monte Carlo Experiments: CCP Estimation

N	K	M	n_{avg}	λ	γ	κ	η	η^e
Population				1.800	0.200	0.800	4.000	5.000
10	80,080	5.0	6.62	1.822	0.202	0.777	4.075	5.072
				(0.005)	(0.001)	(0.012)	(0.236)	(0.235)
12	222,768	8.0	8.29	1.823	0.202	0.775	4.088	5.086
				(0.006)	(0.001)	(0.013)	(0.236)	(0.228)
14	542,640	10.0	9.32	1.823	0.202	0.780	4.076	5.069
				(0.005)	(0.001)	(0.015)	(0.238)	(0.228)
16	1,193,808	13.0	10.89	1.824	0.202	0.787	4.067	5.060
				(0.005)	(0.001)	(0.016)	(0.212)	(0.199)
18	2,422,728	17.0	12.90	1.823	0.202	0.784	4.070	5.064
				(0.006)	(0.002)	(0.014)	(0.210)	(0.189)
20	4,604,600	21.0	14.92	1.822	0.201	0.783	4.061	5.050
				(0.006)	(0.002)	(0.017)	(0.204)	(0.189)

Table: Quality ladder Monte Carlo results: CCP estimation.

Computational Time Comparison

			First	NFXP	CCP		
N	K	Solve v	Stage	Estimation	Setup	Estimation	Total
5	2,310	9.05	0.42	51.61	0.51	0.30	0.81
6	5,544	15.59	0.30	107.41	1.54	0.32	1.86
7	12,012	29.26	0.28	172.40	2.17	0.47	2.64
8	24,024	58.27	0.30	256.19	4.02	0.62	4.64
9	45,045	107.40	0.26	375.26	5.58	1.01	6.59
10	80,080	185.78	0.35	535.83	7.13	1.42	8.55
11	136,136	325.28	0.34	639.98	11.20	1.80	13.00
12	222,768	518.57	0.33	1069.52	13.47	3.21	16.69
13	352,716	821.83	0.34	1411.32	14.96	3.63	18.59
14	542,640	1228.98	0.39	2436.61	17.21	4.10	21.31
15	813,960	1719.72	0.38	3413.42	19.67	7.15	26.82
16	1,193,808	2499.98	0.44	4765.67	23.85	7.06	30.91
17	1,716,099	3642.02	0.43	13513.96	27.28	8.38	35.66
18	2,422,728	5109.30	0.41	10807.54	30.93	10.54	41.47
19	3,364,900	6929.01	0.43	13737.87	35.67	15.37	51.04
20	4,604,600	9377.57	0.41	16069.89	37.26	15.60	52.86

Table: Computational times (in seconds): NFXP vs CCP.

Conclusion

- Our model is empirically tractable and computationally efficient.
- Familiar methods from discrete time, such as two-step estimation, apply almost directly.
- Allows estimation and simulation of counterfactuals using the same model.
- Flexible structure allows for time aggregation, etc.
- Future work: asymmetric information.

REFERENCES

Doraszelski, U. and K. L. Judd (2008). Avoiding the curse of dimensionality in dynamic stochastic games. Unpublished manuscript, Harvard University.

Ericson, R. and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economics and Statistics 62*, 53–82.

Pakes, A. and P. McGuire (1994). Computing markov-perfect nash equilibria: Numerical implications of a dynamic differentiated product model. *RAND Journal of Economics 25*, 555–589.

Pakes, A. and P. McGuire (2001). Stochastic algorithms, symmetric markov perfect equilibrium, and the 'curse' of dimensionality. *Econometrica 69*, 1261–1281.

Rust, J. (1987). Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher. *Econometrica 55*, 999–1013.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008). Markov perfect industry dynamics with many firms. *Econometrica 76*, 1375–1411.