Aradillas-Lopez and Tamer (2007): The Identification Power of Equilibrium in Simple Games

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Introduction

- Examines the identification power of Nash equilibrium assumptions.
- Compare results by dropping Nash equilibrium and using only only *rationalizability*.
- Three examples are considered:
  - 2x2 game of *complete* information (e.g., Bresnahan and Reiss (1991)),
  - 2x2 game of *incomplete* information,
  - First price auction with independent private values.
- Given a random sample, what can we learn about a parameter of interest using only level-k rationalizability.

Equilibrium Concepts

- In simultaneous-move games, players attempt to predict what their rivals will play and act accordingly.
- A **Nash equilibrium** occurs when players' expectations are are consistent with their opponents' actions.
- A **Rationalizable** strategy is a best response to *some* profile of one's opponents' strategies.
- Nash $\subseteq$ Rationalizable.

Behavioral assumptions

- Players use proper subjective probability distributions in analyzing uncertain events,
- Players are expected utility maximizers,
- The rules and structure of the game are common knowledge.

Rationalizability

- A strategy profile for player i is **dominated** if there exists another strategy that is better regardless of what other players do.
- Given a profile of strategies of player i's opponents, s^{-i}, a strategy s^i for player i is a **best response** if it is better than any other strategy given s^{-i}.
- Let $S^i(0)$ denote the set of player i's strategies.
- We say $s^i \in S^i(0)$ is a **level-1 rational strategy** for player i if $\exists s^{-i} \in S^{-i}(0)$ such that s^i is a best response. Let $S^i(1)$ denote the set of such strategies.
- We say $s^i \in S^i(1)$ is a **level-2 rational strategy** for player i if $\exists s^{-i} \in S^{-i}(1)$ such that s^i is a best response.
- And so on. . .

I. 2x2 Game of Complete Information

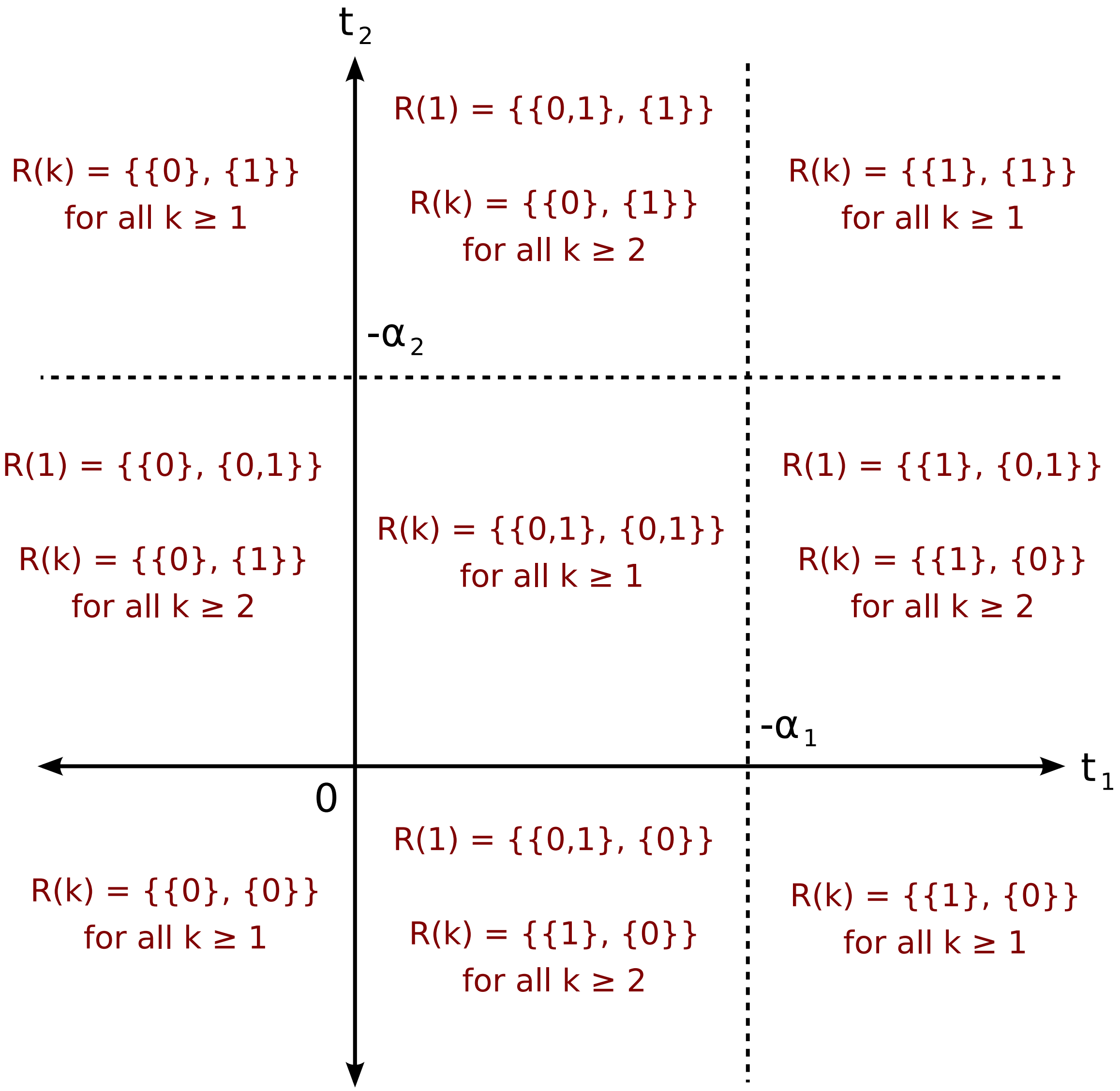
	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$(0,\, 0)$	$(0,\, t_2)$
$a_1 = 1$	$(t_1,\, 0)$	$(t_1 + \alpha_1,\, t_2 + \alpha_2)$

- Standard 2x2 normal-form game.
- Given a sample $\{y_{i1}, y_{i2}\}_{i=1}^{N}$ of market structures (entry decisions) in N independent markets.
- We want to learn about the joint distribution of (t_1, t_2) as well as the parameters α_1 and α_2.
- Assume that $\alpha_1, \alpha_2 \leq 0$.

I. Level-1 rationality

- If $t_1 + \alpha_1 \geq 0$, then $a_1 = 1$ is a dominant strategy for player 1.
- If $t_1 < 0$, $a_1 = 0$, then is a dominant strategy.
- If $t_1 + \alpha_1 \leq 0 \leq t_1$, then both $a_1 = 0$ and $a_1 = 0$ are level-1 rational. They are best responses, respectively, when player 2 plays 1 or 0.
- Similarly for player 2.

I. Level-1 Rationality: Predictions



R(k) = {{0}, {1}}
for all k ≥ 1

R(1) = {{0,1}, {1}}

R(k) = {{0}, {1}}
for all k ≥ 2

R(k) = {{1}, {1}}
for all k ≥ 1

R(1) = {{0}, {0,1}}

R(k) = {{0}, {1}}
for all k ≥ 2

R(k) = {{0,1}, {0,1}}
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R(1) = {{1}, {0,1}}

R(k) = {{1}, {0}}
for all k ≥ 2

R(k) = {{0}, {0}}
for all k ≥ 1

R(1) = {{0,1}, {0}}

R(k) = {{1}, {0}}
for all k ≥ 2

R(k) = {{1}, {0}}
for all k ≥ 1

I. Level-2 rationality

- Consider the region $(t_1, t_2) \in [-\alpha_1, \infty) \times [0, -\alpha_2]$.
- $\mathcal{R}(1) = \{\{1\}, \{0, 1\}\}$
- Player 2 believes player 1 will play $a_1 = 1$ with probability 1.
- $a_2 = 0$ is a best response.
- We can eliminate $a_2 = 1$ at level 2.
- $\mathcal{R}(k) = \{\{1\}, \{0\}\}$ for $k \geq 2$.

I. Inference

- Suppose we know the outcome probabilities $P(0,0),\ldots,P(1,1)$.
- Object of interest: $\theta = (\alpha_1, \alpha_2, F(\cdot,\cdot))$.
- $F(\cdot,\cdot)$ is the joint distribution of $(t_1,\, t_2)$.
- Level-1 rationality implies the following restrictions on θ:

$$\Pr(t_1 \geq -\alpha_1,\, t_2 \geq -\alpha_2) \leq P(1,1) \leq Pr(t_1 \geq 0,\, t_2 \geq 0)$$

$$\Pr(t_1 \leq 0,\, t_2 \leq 0) \leq P(0,0) \leq Pr(t_1 \leq -\alpha_1,\, t_2 \leq \alpha_2)$$

$$\Pr(t_1 \geq -\alpha_1,\, t_2 \leq 0) \leq P(1,0) \leq \Pr(t_1 \geq 0,\, t_2 \leq -\alpha_2)$$

$$\Pr(t_1 \leq 0,\, t_2 \geq -\alpha_2) \leq P(0,1) \leq \Pr(t_1 \leq -\alpha_1,\, t_2 \geq 0)$$

- The identified set Θ_I is the set of all θ which satisfy these inequalities.
- The model *point identifies* θ if Θ_I is a singleton.

II. 2x2 Game of Incomplete Information

	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$(0,\, 0)$	$(0,\, t_2)$
$a_1 = 1$	$(t_1,\, 0)$	$(t_1 + \alpha_1,\, t_2 + \alpha_2)$

- Now assume that t_1 and t_2 are private information.
- Common prior assumption on the joint distribution of $(t_1,\, t_2)$.
- Players have *beliefs* about their opponents' actions, conditional on their own type: $\mathbb{P}_{t_1} \equiv \Pr(a_2 = 1|t_1)$.
- As before, assume that $\alpha_p \leq 0$ for $p = 1, 2$.
- The expected payoffs are now:

$$U(a_i, P_{t_i}) = \begin{cases} t_i + \alpha_i \mathbb{P}_{t_i} & \text{if} \quad a_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

II. Rationality

- Consider only threshold strategies: $Y_p = \mathbf{1}\{t_p \geq \mu_p\}$ for $p = 1, 2$.
- Beliefs are thus probability distributions for μ_{-p} given $\mathcal{I}_p$ (which includes t_p): $\hat{G}_p(\mu_{-p}|\mathcal{I}_p)$.
- The concept of *rationality* now has to account for *level-k rational beliefs*.
- Level-k rationalizable beliefs $\hat{G}_p(\mu_{-p}|\mathcal{I}_p)$ assign zero probability to strictly dominated strategies by player $-p$.
- A strategy by player p is level-k *rationalizable* if it is a best response given level-k rationalizable beliefs:

$$Y_p = \mathbf{1}\left\{t_p + \alpha_p \int_{\mathbb{S}(\hat{G}_p)} \mathrm{E}\left[\mathbf{1}\{t_{-p} \geq \mu\}|\mathcal{I}_p, \mu\right] d\hat{G}_p(\mu|\mathcal{I}_p) \geq 0\right\}$$

- The support $\mathbb{S}(\hat{G}_p)$ of by applying iterated elimination of dominated strategies.

II. Level-1 rationality

- For *any* belief function, the following must hold eventwise:

$$\mathbf{1}\{t_p + \alpha_p \geq 0\} \leq \mathbf{1}\{Y_p = 1\} \quad \text{and} \quad \mathbf{1}\{t_p < 0\} \leq \mathbf{1}\{Y_p = 0\}$$

- All other decision rules are strictly dominated for all possible beliefs.
- The above inequalities imply

$$\Pr(t_p + \alpha_p \geq 0) \leq \Pr(t_p \geq \mu_p) \leq \Pr(t_p \geq 0)$$

  or simply, $\mu_p \in [0, -\alpha_p]$.

- This is the set of *level-1 rationalizable strategies.*

II. Level-2 rationality

- Level 2 rationalizable beliefs:
  - assign zero probability to strictly dominated strategies $\mu_p \notin [0, -\alpha_p]$,
  - satisfy $\hat{G}(0|\mathcal{I}_p) = 0$ and $\hat{G}(-\alpha_p|\mathcal{I}_p) = 1$.
- Level 2 rationalizable strategies are:
  - level 1 rationalizable (i.e., $0 \leq \mu_p \leq -\alpha_p$),
  - best responses given level-2 rational beliefs.
- A strategy $Y_p = \mathbf{1}\{t_p \geq \mu_p\}$ is *level-2 rationalizable* if

$$\mu_p = -\alpha_p \int_0^{-\alpha_{-p}} \mathrm{E}\left[\mathbf{1}\{t_{-p} \geq \mu\}|\mathcal{I}_p, \mu\right] \, d\hat{G}_p(\mu|\mathcal{I}_p)$$

II. Level-k rationality

We can summarize the set of level-k rationalizable strategies in the class of threshold strategies $Y_p = \mathbf{1}\{t_p \geq \mu_p\}$ as follows:

- For $k = 1$ and $p \in \{1, 2\}$,

$$\mu_p \in [\mu^L_{p,k},\ \mu^U_{p,k}] \equiv [0, -\alpha_p]$$

- For $k > 1$ and $p \in \{1, 2\}$,

$$\mu_p \in [\mu^L_{p,k},\ \mu^U_{p,k}] \equiv \Big[-\alpha_p \, \mathrm{E}\left[\mathbf{1}\{t_{-p} \geq \mu^U_{-p,k-1}\}|\mathcal{I}_p\right], \\ -\alpha_p \, \mathrm{E}\left[\mathbf{1}\{t_{-p} \geq \mu^L_{-p,k-1}\}|\mathcal{I}_p\right]\Big]$$

II. A Parametric Model

- Let $t_p = X_p^\top \beta_p - \varepsilon_p$ for $p \in \{1, 2\}$ where X_p is observable to the researcher but ε_p is not. We wish to estimate β_p.
- For simplicity, ε_1 and ε_2 are independent and $\varepsilon_p \sim H_p(\cdot)$.
- Player p knows ε_p and believes $\varepsilon_{-p} \sim H_{-p}(\cdot)$.
- We now proceeds as follows:
  1. Develop an objective function which can be used to construct the identified set Θ_I,
  2. Discuss identification of k,
  3. Provide sufficient conditions for point identification.

II. Iterative Construction of Beliefs

- We iteratively construct bounds on the beliefs $\Pr(Y_{-p}|\mathcal{I})$:

$$\pi^L_{-p}(\theta|k,\mathcal{I}) \leq \Pr(Y_{-p}|\mathcal{I}) \leq \pi^U_{-p}(\theta|k,\mathcal{I}).$$

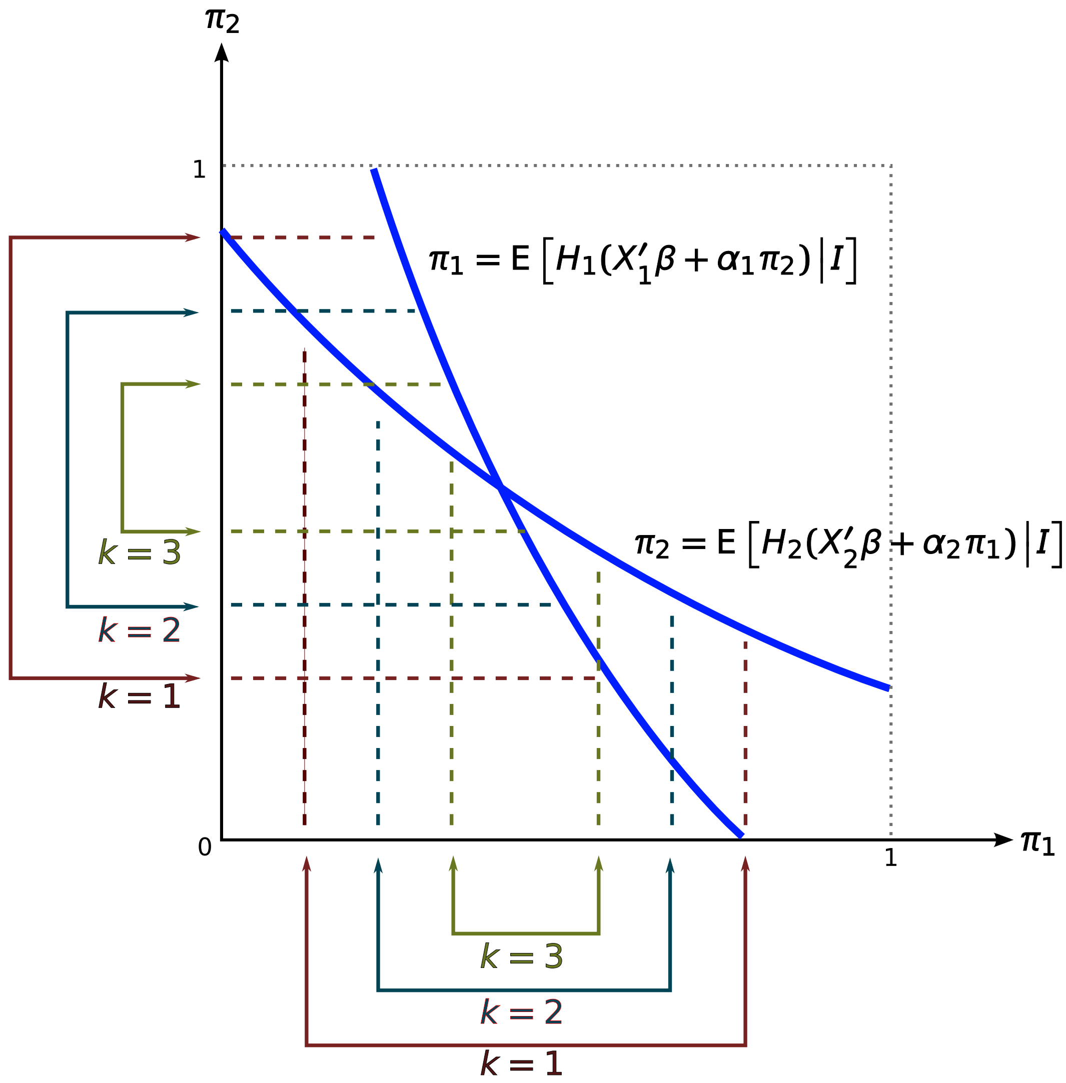
- Initialize $\pi^L_{-p}(\theta|k=1,\mathcal{I}) = 0$ and $\pi^U_{-p}(\theta|k=1,\mathcal{I}) = 1$.
- Then for each $k > 1$ and each $p \in \{1,2\}$:

$$\begin{aligned} \pi^L_p(\theta|k,\mathcal{I}) &= H_p\left(X_p^\top \beta_p + \alpha_p \pi^U_{-p}(\theta|k-1,\mathcal{I})\right), \\ \pi^U_p(\theta|k,\mathcal{I}) &= H_p\left(X_p^\top \beta_p + \alpha_p \pi^L_{-p}(\theta|k-1,\mathcal{I})\right). \end{aligned}$$

- It follows that

$$\left[\pi^L_{-p}(\theta|k,\mathcal{I}), \pi^U_{-p}(\theta|k,\mathcal{I})\right] \subseteq \left[\pi^L_{-p}(\theta|k-1,\mathcal{I}), \pi^U_{-p}(\theta|k-1,\mathcal{I})\right] \quad \text{a.s.}$$

II. Iterative Belief Construction Example



II. Finding the Identified Set

- Player p is k-rational if and only if

$$\mathbf{1}\left\{X_p^\top \beta_p + \alpha_p \pi^U_{-p}(\theta|k,\mathcal{I}) \geq \varepsilon_p\right\} \leq \mathbf{1}\left\{Y_p = 1\right\} \leq \mathbf{1}\left\{X_p^\top \beta_p + \alpha_p \pi^L_{-p}(\theta|k,\mathcal{I}) \geq \varepsilon_p\right\}$$

- We can use this relationship as a basis for inference.
- Given some k, let $W_p \equiv (X_p, \mathcal{I})$ and let $a, b \in \mathbb{R}^{\dim(W_p)}$:

$$\Lambda_p(\theta|a,b,k) = \mathrm{E}\left[\left(1 - \mathbf{1}\left\{\begin{array}{c} H_p\left(X_p^\top \beta_p + \alpha_p \pi^U_{-p}(\theta|k,\mathcal{I})\right) \\ \leq \Pr(Y_p = 1|W_p) \\ \leq H_p\left(X_p^\top \beta_p + \alpha_p \pi^L_{-p}(\theta|k,\mathcal{I})\right) \end{array}\right\}\right) \mathbf{1}\left\{a \leq W_p \leq b\right\}\right].$$

- Then define

$$\Gamma_p(\theta|k) = \iint \Lambda_p(\theta|a,b,k)\, dF_{W_p}(a)\, dF_{W_p}(b)$$

and

$$\Gamma(\theta|k) = \left(\Gamma_1(\theta|k),\, \Gamma_2(\theta|k)\right)^\top$$

II. Finding the Identified Set

- We can use the previous expression to construct an objective function.
- For some positive definite matrix Ω,

$$\Theta(k) = \left\{ \theta \in \arg\min_{\theta} Q(\theta|k) \equiv \arg\min_{\theta} \Gamma(\theta|k)^{\top} \Omega \Gamma(\theta|k) \right\}.$$

- By construction, $\Theta(k+1) \subseteq \Theta(k)$ for all k.
- Given a random sample, we can use set inference methods to construct an estimator for $\Theta(k)$.
- As compared with Nash equilibrium, we do not have to solve a fixed point problem. The iterative restrictions imposed by rationality are computationally simple by comparison.
- For any k, $\Theta(k)$ is guaranteed to contain the BNE.

II. Inference on the Rationality Level

- A sample can also inform us about the level of rationality k.
- Suppose all players are at most k_0-rational.
- The level-k_0 bounds should hold *a.e.* but the level-$(k_0 + 1)$ bounds may be violated.
- We can proceed as follows:
  - Construct $\Theta(1)$,
  - Define $\tilde{Q}(k) = \min_{\theta \in \Theta(1)} Q(\theta|k)$,
  - $\tilde{Q}(k) = 0$ for $k \leq k_0$ but $\tilde{Q}(k) > 0$ if $k > k_0$.
- Thus, if $\tilde{Q}(k) > 0$ and $\tilde{Q}(k-1) = 0$ we can reject $k_0 = k$.

II. Point Identification Under Level-1 Rationality

- Suppose players are level-1 rational.
- Suppose X_p has full rank for $p \in \{1,2\}$. Let $X_{d,p}$ and $X_{l,p}$ denote, respectively, regressors with bounded and unbounded support and let $\beta_{d,p}$ and $\beta_{l,p}$ be the corresponding coefficients. Finally, define $X \equiv (X_1, X_2)$.
- $\beta_{l,p}$ is identified if for each p, there is a continuous $X_{l,p}$ with nonzero $\beta_{l,p}$ and unbounded support conditional on $X_{-l,p}$ such that for any $c \in (0,1)$, $b \neq 0$ and $q \in \mathbb{R}^{\dim(X_{-l,p})}$, there exists $C_{b,q,m} > 0$ such that

$$\Pr(\varepsilon_p \leq bX_{l,p} + q^\top X_{-l,p} | X) > m$$

for all $X_{l,p}$ and $\operatorname{sgn}(b) \cdot X_{l,p} > C_{b,q,m}$.

II. Point Identification Under Level-1 Rationality

- $\beta_{d,p}$ is identified if for any $\alpha_p, \beta_{d,p}, \tilde{\beta}_{d,p} \in \Theta$ with $\tilde{\beta}_{d,p} \neq \beta_{d,p}$,

$$\Pr\left(\left|X_{d,p}^{\top}(\beta_{d,p} - \tilde{\beta}_{d,p})\right| > |\alpha_p| \,\middle|\, X_{-d,p}\right) > 0.$$

- If the above two properties hold, and for any $\Delta > 0$, there exists $\mathcal{X}_\Delta \in \mathbb{S}(X_p)$ such that $\Pr(Y_p = 1|X) < \Pr(\varepsilon_p \leq X_p^{\top}\beta_{b_0} + \alpha_{p_0}|X) + \Delta$ whenever $X_p \in \mathcal{X}_\Delta$, then the identified set for α_p is $\{\alpha_p \leq \alpha_{p_0}\}$.

III. First Price IPV Auction

- Many symmetric, risk-neutral potential buyers.
- Bids are made simultaneously for a single good.
- Focus on independent private values (IPV) v_i.
- Object of interest: $F_0(\cdot)$, the distribution of private values.
- $F_0(\cdot)$ is common knowledge among all players; has support $[0, \omega)$.
- For simplicity, assume the reserve price is $p_0 = 0$.

III. Interim Rationality

- Point identification in BNE case established by Guerre, Perrigne and Vuong (2000).
- Here, the equilibrium assumption is relaxed following Battigalli and Siniscalchi's (2003).
- Buyers under *interim rationality*:
  - rational,
  - expected utility maximizers,
  - strategically sophisticated to a particular degree k,
  - have beliefs that may or may not be "correct."

III. Assumptions on Beliefs

- Bidders expect that *any* positive bid will win with positive probability:
  - No player will bid higher than v_i,
  - Each bidder with $v_i > 0$ will submit a strictly positive bid.
  - The number of potential bidders $\mathcal{N}$ equals the number of actual bidders *a.s.*
- Beliefs only assign positive probability to *increasing* bidding functions.
- The space of all such functions is

$$\mathcal{B} = \{b(\cdot) : [0, \omega) \rightarrow \mathbb{R}_+ | b(v) \leq v \quad \text{and} \quad v > w \Rightarrow b(v) > b(w)\}.$$

III. Level-1 Rationality

- Player i's problem is

$$\max_{b\geq 0}(v_i - b)\hat{\Pr}_i\left[\max_{j\neq i} b(v_j) \leq b\right].$$

- Here, $\hat{\Pr}_i$ denotes player i's beliefs about v_{-i}.
- Level-1 rational bid satisfy

$$b \leq v_i \equiv \bar{B}_1(v_i, \mathcal{N}).$$

- Any bid show *any* bid below v_i is level-1 rational (and thus we cannot bound bids from below).

III. Level-2 Rationality

- The worst case scenario for player i is that for all j,

$$b(v_j) = \bar{B}_1(v_j, \mathcal{N}) = v_j.$$

- Expected utility in this case is

$$\max_{b \geq 0}(v_i - b)\hat{\Pr}_i\left[\max_{j \neq i} \bar{B}_1(v_j, \mathcal{N}) \leq b\right] = \max_{b \geq 0}(v_i - b)F_0(b)^{\mathcal{N}-1} \equiv \pi_2(v_i, \mathcal{N}).$$

- $\pi_2(v_i, \mathcal{N})$ is thus a lower bound over all beliefs. The upper bound is $(v_i - b)$.
- Thus, rational bids must satisfy

$$v_i - b \geq \pi_2(v_i, \mathcal{N})$$

or

$$b \leq v_i - \pi_2(v_i, \mathcal{N}) \equiv \bar{B}_2(v_i, \mathcal{N}).$$

III. Level-k Rationality

- Proceeding recursively, one can show that at level-k,

$$b_i \leq v_i - \pi_k(v_i, \mathcal{N}) \equiv \bar{B}_k(v_i, \mathcal{N})$$

where

$$\pi_k(v_i, \mathcal{N}) = \max_{b \geq 0}(v_i - b) F_0\left(\bar{S}_{k-1}(b, \mathcal{N})\right)^{\mathcal{N}-1}$$

and

$$\bar{S}_{k-1}(b, \mathcal{N}) \equiv \bar{B}_k^{-1}(\cdot, \mathcal{N}).$$

- We know that $\bar{B}_k(\cdot, \mathcal{N}) \geq b^{BNE}(\cdot, \mathcal{N})$ for all k. Furthermore, bidding below b^{BNE} is rationalizable for all k.
- That is, the predictive power of k-rationality in this model is significantly less than that of BNE.

III. Identification with Level-1 Rationality

- Suppose F_0 lies in the space of log-concave, absolutely continuous distribution functions on $[0, \omega)$.
- We are given a sample of L auctions and want to recover F.
- The model predicts that level-1 rational bids satisfy $0 \leq b_i \leq v_i$ for all $i = 1, \ldots, \mathcal{N}$.
- The b_i's are observed but the v_i's are not, but we can still bound F:

$$F_0(t, \theta) \equiv \Pr(v \leq t) \leq \Pr(b \leq t) \equiv G_b(t).$$

- Empirical strategy: bound the distribution of valuations using the empirical distribution of bids.

III. Identification with Level-k Rationality

- Continue for $k > 1$ using the bounds derived above.
- We can derive an objective function as before:

$$\Lambda(\theta|a,c,k) = \int \left(1 - \mathbf{1}\left\{F_b(b) \geq F\left(\bar{S}_k(b,\mathcal{N}|\theta),\theta\right)\right\}\right)\mathbf{1}\left\{a \leq b \leq c\right\}\, dF_b(b)$$

$$\Lambda(\theta|k) = \iint \Lambda(\theta|a,c,k)\, dF_b(a)\, dF_b(c)$$

$$\Theta(k) = \left\{\theta \in \Theta : \theta \in \arg\min_{\theta\in\Theta} \Gamma(\theta|k)^2\right\}$$

Conclusion

- Look at identification power of Nash equilibrium by comparing to level-k rationality.
- Derivation of identified set and corresponding objective functions for conducting inference.
- Three cases:
  - 2x2 game of complete information,
  - 2x2 game of incomplete information,
  - first price auction.