Solving Nonlinear Wave Equations with Mathematica

Jason R. Blevins

Tuesday, April 22, 2003

Dr. Willy A. M. Hereman
Colorado School of Mines

Ünal Göktaş (Wolfram Research, Inc.)
Ryan Martino, Joel Miller, Linda Hong (REU '99)
Doug Baldwin, Steve Formaneck, Andrew Menz (REU '00)
Doug Baldwin, Ben Kowalski (REU '01)
Jeff Heath, Paul Blanchard, Jack Milwid,
Matthew Porter-Peden (REU '02)

Supported in part by NSF Grants DMS-9912293
and CCR-9901929

Background

- Traveling wave solutions of nonlinear PDEs and ODEs
- $Tanh$ and $Sinh$ solutions
- $Tanh$ algorithm
- Symbolic algorithms to compute and verify exact solutions
- PDESpecialSolutions
- PDESolutionTester

Project Overview

I. PDESpecialSolutions

- Detailed Debugging Code
- Extensive Testing
- Implementation Improvement

II. PDESolutionTester

- Algorithm Design
- Initial Implementation
- Preliminary Testing
- Extend and Improve Implementation

Tanh Algorithm: Step T1

Transform the PDE into a nonlinear ODE

- Given: System of nonlinear PDEs of order m with M dependent variables $(u, v, w, ...)$ and N independent variables $(x, y, z, ..., t)$.

- Seek solutions polynomial in T, with

$$T = \tanh \xi = \tanh \left[\sum_{j}^{N} c_j x_j + \Delta \right].$$

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, *all* derivatives of T are polynomial in T. For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dT} \frac{\partial T}{\partial x_j} = c_j (1 - T^2) \frac{d \bullet}{dT}.$$

Boussinesq System: Step T1

- Boussinesq system:

$$
\begin{aligned}
u_t + v_x &= 0 \\
v_t + u_x - 3uu_x - \alpha u_{3x} &= 0
\end{aligned}
$$

- Cancel common factors of $1 - T^2$,

$$
\begin{aligned}
c_2 U' + c_1 V' &= 0 \\
c_2 V' + c_1 U' - 3c_1 U U' & \\
+\alpha c_1^3 \left[2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2 U'''\right] &= 0
\end{aligned}
$$

Tanh Algorithm: Step T2

- Seek polynomial solutions

$$U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j.$$

- Determine the highest exponents $M_i \geq 1$.
- Substitute $U_i(T) = T^{M_i}$ into the LHS of ODE.
- Gives polynomial $\mathbf{P}(T)$.
- For every P_i consider all possible balances of the highest exponents in T.
- Solve the resulting linear system(s) for the unknowns M_i.

Boussinesq System: Step T2

- Balance highest exponents for the Boussinesq system:

$$M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1$$

- So, $M_1 = M_2 = 2$

- Hence,

$$\begin{aligned} U(T) &= a_{10} + a_{11}T + a_{12}T^2 \\ V(T) &= a_{20} + a_{21}T + a_{22}T^2 \end{aligned}$$

Tanh Algorithm: Step T3

- Derive algebraic system for the unknown coefficients a_{ij} by setting the coefficients of the power terms in T to zero.

- Example: Algebraic system for Boussinesq case

$$
\begin{aligned}
a_{11}c_1(3a_{12} + 2\alpha c_1^2) &= 0 \\
a_{12}c_1(a_{12} + 4\alpha c_1^2) &= 0 \\
a_{21}c_1 + a_{11}c_2 &= 0 \\
a_{22}c_1 + a_{12}c_2 &= 0 \\
a_{11}c_1 - 3a_{10}a_{11}c_1 + 2\alpha a_{11}c_1^3 + a_{21}c_2 &= 0 \\
-3a_{11}^2c_1 + 2a_{12}c_1 - 6a_{10}a_{12}c_1 & \\
+16\alpha a_{12}c_1^3 + 2a_{22}c_2 &= 0
\end{aligned}
$$

Tanh Algorithm: Step T4

- Solve the nonlinear algebraic system with parameters.

- Example: Solution for Boussinesq system

$$
\begin{aligned}
a_{10} &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} \\
a_{11} &= 0 \\
a_{12} &= -4\alpha c_1^2 \\
a_{20} &= \text{free} \\
a_{21} &= 0 \\
a_{22} &= 4\alpha c_1 c_2
\end{aligned}
$$

Tanh Algorithm: Step T5

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.

- Example: Solitary wave solution for Boussinesq system:

$$
\begin{aligned}
u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[c_1 x + c_2 t + \Delta\right] \\
v(x,t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2\left[c_1 x + c_2 t + \Delta\right]
\end{aligned}
$$

PDESolutionTester Algorithm

Example: For the Boussinesq system,

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

with $PDESpecialSolutions$ (Tanh option) we found the solution

$$\begin{aligned} u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[c_1 x + c_2 t + \Delta\right], \\ v(x,t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2\left[c_1 x + c_2 t + \Delta\right]. \end{aligned}$$

For this example, we will use $PDESolutionTester$ to test solutions of the form

$$\begin{aligned} u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - b \tanh^2\left[c_1 x + c_2 t\right], \\ v(x,t) &= a_{20} + d \tanh^2\left[c_1 x + c_2 t\right]. \end{aligned}$$

Solution Tester: Step 1

Apply Taylor Approximations

- Substitute taylor expansions for all elementary functions
- Substitute the solutions back into the original system
- Example: The Boussinesq system has order 3. Substitute a taylor polynomial of order 7.

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315}.$$

Solution Tester: Step 2

Find Constraints

- Choose an independent variable, set others to zero
- Homogeneous order by order system of coefficients of chosen independent variable
- Iterate over all other independent variables, combine resulting systems
- The nonlinear algebraic solver *AnalyzeAndSolve* solves the system
- Example: For the Boussinesq equations, the software returns $b = 4\alpha c_1^2$ and $d = 4\alpha c_1 c_2$, thus yielding the correct solution set:

$$
\begin{aligned}
u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - b \tanh^2\left[c_1 x + c_2 t\right] \\
v(x,t) &= a_{20} + d \tanh^2\left[c_1 x + c_2 t\right]
\end{aligned}
$$

Solution Tester: Step 3

Verify Solutions

1. Numeric Test
   - Apply newly found constraints
   - Apply 13 sets of random numbers
   - If any of 13 less than 10^{-10}, solution passes
   - Around 1 in 10000 chance of missing a solution
2. Symbolic Test
   - Apply newly found constraints
   - Replace trig functions by exponentials
   - Simplify, Factor, Simplify, etc...
   - Try to simplify to zero

Conclusion

- Dr. Willy A. M. Hereman
  http://www.mines.edu/fs_home/whereman/

- REU info:
  http://www.nsf.gov/home/crssprgm/reu/