Lecture 10: $t$-Tests

1. The $t$-Test

William Gosset, a chemist at the Guinness brewery in Dublin, introduced the $t$ distribution in an 1908 article published in the journal *Biometrika* under the pen name “Student” (Student, 1908). This paper was a result of Gosset's work in applying statistics to help Guinness select the best yielding varieties of barley. He was forced to use a pen name because, at the time, Guinness regarded the fact that they were using statistics as a trade secret. Even fellow statisticians didn't know Gosset's identity.

A $t$-test is a hypothesis test involving a test statistic which follows Student's $t$ distribution under the null hypothesis. We can use a $t$-test to test hypotheses about single slope coefficients $\beta_k$ for regressors $k = 1, \ldots, K$ in the linear model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_K X_{Ki}.$$  

They can be applied to both two-sided hypotheses such as

$$H_0 : \beta_k = \beta_{H_0}$$

$$H_A : \beta_k \neq \beta_{H_0}$$

and one-sided hypotheses such as

$$H_0 : \beta_k \geq \beta_{H_0}$$

$$H_A : \beta_k < \beta_{H_0}$$

or

$$H_0 : \beta_k \leq \beta_{H_0}$$

$$H_A : \beta_k > \beta_{H_0}$$

The value $\beta_{H_0}$ in these examples is called the border value and is the value in the null hypothesis that is closest to the rejection region.

The $t$-statistic for a null hypothesis with border value $\beta_{H_0}$

$$t_k = \frac{\hat{\beta}_k - \beta_{H_0}}{\text{SE}(\hat{\beta}_k)}.$$  

The most common $t$-test is for the null hypothesis $H_0 : \beta_k = 0$, where $\beta_{H_0} = 0$. In such cases, the $t$-statistic simplifies to

$$t_k = \frac{\beta_k}{\text{SE}(\hat{\beta}_k)}.$$
When the \( t \) statistic is larger in absolute value, it means that there is a greater likelihood that the estimated coefficient is statistically significantly different from zero.

Student’s \( t \) distribution has one parameter: the number of degrees of freedom. In the context of an ordinary least squares regression with \( K \) regressors and \( n \) observations, there are \( n - K - 1 \) degrees of freedom. Since this is a well-known distribution, the critical values for tests of common sizes (e.g., \( \alpha = 0.10, 0.05, 0.01 \)) have been tabulated.

To find the critical value, denoted \( t_c \), for a given \( t \)-test, one needs to know the degrees of freedom, the level of the test, and whether the test is one-sided or two sided. Given these pieces of information, the critical value can be found in a table such as Table B-1 of Studenmund (2010).

Once you know the value of the \( t \) statistic, \( t_k \), and the critical value, \( t_c \), the decision to reject or fail to reject is determined as follows:

- **Reject** \( H_0 \) if \( |t_k| > t_c \) and if \( t_k \) has the sign corresponding to \( H_A \).
- **Fail to reject** \( H_0 \) otherwise.

Note that with a two-sided test, the sign test can be skipped since the alternative hypothesis specifies both signs.

A common rule of thumb for testing \( H_0 : \beta_k = 0 \) against \( H_A : \beta_k \neq 0 \) at the 5% level is: reject \( H_0 \) if \( \hat{\beta}_k / \text{SE}(\hat{\beta}_k) \) exceeds 2 in absolute value. Stated differently, we reject \( H_0 \) if \( \hat{\beta}_k > 2 \cdot \text{SE}(\hat{\beta}_k) \). This rule is derived from the fact that the critical value for the two sided, 5% \( t \)-test is 1.96 \( \approx \) 2 when the number of observations is large. This rule of thumb is useful for quickly looking at a table of regression results (estimates and standard errors) and carrying out 5% significance tests for each coefficient.

Finally, it is important to note the limitations of \( t \)-tests. They do not test for the theoretical validity of explanatory variables nor do they test for the importance of those variables.

### 2. Confidence Intervals

A **confidence interval** is an interval of values for a coefficient \( \beta_k \) that contains the true value a fixed percentage of the time. A confidence interval with confidence probability \( 1 - \alpha \) for \( \beta_k \) can be constructed using the standard error, \( \text{SE}(\hat{\beta}_k) \), and the critical value for the two-sided \( t \) test of level \( \alpha \), \( t_c \), as follows:

\[
\text{CE} = \hat{\beta}_k \pm t_c \cdot \text{SE}(\hat{\beta}_k).
\]

That is, the left endpoint of the interval is given by \( \hat{\beta}_k - t_c \cdot \text{SE}(\hat{\beta}_k) \) and the right endpoint is given by \( \hat{\beta}_k + t_c \cdot \text{SE}(\hat{\beta}_k) \). We say the interval \( \text{CE} \) is a \( 1 - \alpha \) percent confidence interval for \( \beta_k \).

There is a close relationship between confidence intervals and two-sided \( t \) tests. If the “border value” \( \beta_{H_0} \) falls inside the confidence interval, then we cannot reject \( H_0 \) at the \( \alpha \) level.

To make this more concrete, suppose that \( \text{CE}_{1-\alpha} \) is a 95% confidence interval for \( \beta_1 \), with \( \alpha = 0.05 \) and we are testing the null hypothesis \( H_0 : \beta_1 = 0 \). If the interval \( \text{CE}_{1-\alpha} \) contains 0, then we cannot reject \( H_0 \) at level \( \alpha \). If 0 falls outside of the interval, then we can reject \( H_0 \).
3. Example: Financial Aid

Consider the regression model

\[ \text{FINAID}_i = \beta_0 + \beta_1 \text{HSRANK}_i + \beta_2 \text{PARENT}_i + \beta_3 \text{MALE}_i + \epsilon_i \]

where \( \text{FINAID}_i \) is the amount of financial aid awarded to student \( i \) at a small liberal arts school, \( \text{HSRANK}_i \) is the student’s GPA rank in high school (measured as a percentage from 0 to 100), \( \text{PARENT}_i \) is the expected financial contribution of the student’s parents (a measure of financial need), and \( \text{MALE}_i \) is an indicator variable which equals one if the student is male.

First, we must state a hypothesis for each variable.

1. \( H_0: \beta_1 \leq 0 \) vs \( H_A: \beta_1 > 0 \)
2. \( H_0: \beta_2 \geq 0 \) vs \( H_A: \beta_2 < 0 \)
3. \( H_0: \beta_3 = 0 \) vs \( H_A: \beta_3 \neq 0 \)

Then, with our sample of \( n = 50 \) observations, we obtain the estimates given in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>9813.02</td>
<td>1743.10</td>
</tr>
<tr>
<td>HSRANK</td>
<td>83.2612</td>
<td>20.1479</td>
</tr>
<tr>
<td>PARENT</td>
<td>-0.342754</td>
<td>0.0315054</td>
</tr>
<tr>
<td>MALE</td>
<td>-1570.14</td>
<td>784.297</td>
</tr>
</tbody>
</table>

Table 1. Financial aid

Using these results, we can choose a level \( \alpha \) and test each of the hypotheses. Suppose we choose \( \alpha = 0.05 \).

1. \( H_0: \beta_1 \leq 0 \) vs \( H_A: \beta_1 > 0 \) is a one-sided test. The \( t \)-value for this test is
\[ t_1 = \frac{83.2612}{20.1479} = 5.630. \]

The critical value for a one-sided, 5% \( t \)-test with \( n - K - 1 = 50 - 3 - 1 = 46 \) degrees of freedom is
\[ t_c = 1.684. \]

Since \( |t_1| = 5.630 > t_c = 1.684 \) and the signs of \( t_1 \) and \( H_A \) agree, we reject \( H_0 \).

2. \( H_0: \beta_2 \geq 0 \) vs \( H_A: \beta_2 < 0 \) is again a one-sided test. The \( t \)-value for this test is
\[ t_2 = \frac{-0.342754}{0.0315054} = -10.879. \]

The critical value is the same as before: \( t_c = 1.684 \). Since \( |t_2| = 10.879 > 1.684 \) and since \( t_2 \) is negative, corresponding to \( H_A \), we again reject \( H_0 \).

3. \( H_0: \beta_3 = 0 \) vs \( H_A: \beta_3 \neq 0 \) is a two-sided test so we must use the corresponding two-sided critical value: \( t_c = 2.021 \). The test statistic in this case is
\[ t_3 = \frac{-1570.14}{784.297} = -2.002. \]

and since \( |t_3| < 2.021 \), we cannot reject \( H_0 \).
4. Example: 401(k) Participation

Recall our model of employee participation in employer 401(k) plans. The participation rate, $PRATE_i$, is a function of the matching rate $MRATE_i$ and the age of the plan $AGE_i$. We estimated the regression equation

$$PRATE_i = \beta_0 + \beta_1 MRATE_i + \beta_2 AGE_i + \varepsilon_i$$

with $n = 1000$ observations and found the results given in Table 2.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>S.E.</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.</td>
<td>80.1922</td>
<td>0.985611</td>
<td>81.363</td>
</tr>
<tr>
<td>MRATE</td>
<td>5.2946</td>
<td>0.639931</td>
<td>8.274</td>
</tr>
<tr>
<td>AGE</td>
<td>0.2641</td>
<td>0.059008</td>
<td>4.476</td>
</tr>
</tbody>
</table>

Table 2. Participation in 401(k) plans

Given the estimates and standard errors, we could have calculated the values of the $t$ statistic for $H_0 : \beta_k = 0$ for each of the coefficients given above. For example, for $\beta_1$ we have

$$t_1 = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)} = \frac{80.1922}{0.985611} = 81.363.$$  

In this application, the number of degrees of freedom is $n - K - 1 = 1000 - 2 - 1 = 997$, so we can use the critical values for $\infty$.

Using the rule of thumb $t_k > 2$, we can easily reject the null hypothesis $H_0 : \beta_k = 0$ for each slope coefficient at the 5% level.

We might also be interested in testing the following one-sided tests:

- Carry out the following test with $\alpha = 0.01$:
  $$H_0 : \beta_2 \leq 0$$  
  $$H_A : \beta_2 > 0$$

  The critical value for this one-sided test is $t_c = 2.326$. Since $|t_1| = 8.274 > 2.326$ and since the signs of $t_1$ and $H_A$ agree, we reject $H_0$.

- Carry out the following test with $\alpha = 0.10$:
  $$H_0 : \beta_1 \leq 0$$  
  $$H_A : \beta_1 > 0$$

  The critical value for this one-sided test is $t_c = 1.282$. Since $|t_2| = 4.476 > 1.282$ and since the signs of $t_2$ and $H_A$ agree, we reject $H_0$.

References
