Identification and Estimation of Continuous Time Dynamic Discrete Choice Games

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Overview

Paper concerns continuous time dynamic discrete choice games:

- Generalizes Arcidiacono, Bayer, Blevins, and Ellickson (2016).
- Infinite horizon game, time indexed by $t \in [0, \infty)$.
- Firms $i = 1, \ldots, N$ maximize expected discounted profits.
- Finite states $k = 1, \ldots, K$.
- Discrete actions w/choice-specific errors as in DT.
- Decision times not fixed, but $\sim$ Exponential(λ_{ik}).
- Markov perfect equilibrium in choice probabilities.
- CT reduced form: choice-specific hazards.
- DT reduced form: state-to-state transition probabilities.

Contributions of This Paper

1. Identification of heterogeneous rates λ_{ik}:
   - Previous work assumed $\lambda = 1$.
   - Can we identify and estimate the rates λ?
   - What about heterogeneous rates λ_{ik}?
   - Firm-state-specific rates allow strategic differences.
   - Firm heterogeneity may reduce multiplicity.
2. Re-establish some important theoretical properties:
   - Existence of a Markov perfect equilibrium (MPE).
   - Linear representation of value function given conditional choice probabilities (CCPs).
3. Identification with only DT "snapshot" data:
   - Identification of CT reduced form from DT data.
   - Identification of structural primitives from CT reduced form.
4. Empirical and Monte Carlo evidence with canonical examples:
   - Single agent renewal model.
   - Dynamic oligopoly quality ladder model.

Motivation: Computational Advantages

- Estimation of dynamic discrete choice games is difficult.
  - Full-solution (NFXP) following Rust (1987) was infeasible.
  - Two-step (CCP) estimation proved useful (Rust, 1994, Aguirregabiria and Mira, 2007, Bajari, Benkard, and Levin, 2007, Pakes, Ostrovsky, and Berry, 2007).
- These allow us to *estimate* complex games, but *solving and simulating* them remains difficult.
- Hard to handle more than a few firms and one state variable.
- Computational complexity in DT of firms' expectations:
  - Suppose N players can each move to one of κ states.
  - Due to *simultaneity*, firms have beliefs about κ^N future states.
- Researchers forgo counterfactuals or use simpler models.
- In CT, firms consider $N(\kappa - 1)$ state changes (linear in N).

Motivation: Economic Implications

- Often, data are "snapshots" at equispaced intervals.
- Discrete time, simultaneous-move models match this.
- But this is a sampling limitation, not necessarily a desirable model feature.
- "Simultaneous move" paradigm has both informational and timing implications.
- Specifying simultaneous moves when they are sequential leads to bias in entry costs, competitive effects, etc.
- Instead, we specify the model at the level of real actions and aggregate to the data frequency for estimation.

Previous and Related Work

1. Doraszelski and Judd (2012)
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
3. Blevins (2017)
4. Blevins and Kim (2024)
5. Blevins (2025) (WP)
6. Applications of continuous time models

Previous and Related Work

1. Doraszelski and Judd (2012)
   - Theoretical model for continuous time dynamic games.
   - Computational advantage of sequential state changes.
   - Firms have beliefs about only $(\kappa - 1)N$ future states.
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
3. Blevins (2017)
4. Blevins and Kim (2024)
5. Blevins (2025) (WP)
6. Applications of continuous time models

Previous and Related Work

1. Doraszelski and Judd (2012)
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
   - Empirical model with multinomial choice structure.
   - Inverse CCP representation as in Hotz and Miller (1993).
   - Unobserved heterogeneity as in Arcidiacono and Miller (2011).
   - Estimate effects of Walmart entering retail grocery industry.
3. Blevins (2017)
4. Blevins and Kim (2024)
5. Blevins (2025) (WP)
6. Applications of continuous time models

Previous and Related Work

1. Doraszelski and Judd (2012)
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
3. Blevins (2017)
   - Identification of CT model from DT data (Phillips, 1972).
   - Rank condition based on smaller number of prior restrictions.
4. Blevins and Kim (2024)
5. Blevins (2025) (WP)
6. Applications of continuous time models

Previous and Related Work

1. Doraszelski and Judd (2012)
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
3. Blevins (2017)
4. Blevins and Kim (2024)
   - CT version of NPL estimator (Aguirregabiria and Mira, 2007).
   - Consistent & asymptotically normal iterative estimator.
5. Blevins (2025) (WP)
6. Applications of continuous time models

Previous and Related Work

1. Doraszelski and Judd (2012)
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
3. Blevins (2017)
4. Blevins and Kim (2024)
5. Blevins (2025) (WP)
    - Contractivity of value iteration for fixed policies.
    - "Uniform" representation connecting CT and DT models.
    - Newton-Kantorovich iterations for solving equilibrium.
    - Analytical derivatives of matrix exponential (log likelihood).
6. Applications of continuous time models

Previous and Related Work

1. Doraszelski and Judd (2012)
2. Arcidiacono, Bayer, Blevins, and Ellickson (2016)
3. Blevins (2017)
4. Blevins and Kim (2024)
5. Blevins (2025) (WP)
6. Applications of continuous time models
   - Airlines and rail in China: Qin, Vitorino, and John (2022)
   - Movie theaters: Takahashi (2015)
   - Allocation of donor kidneys: Agarwal, Ashlagi, Rees, Somaini, and Waldinger (2021)
   - Online gaming: Nevskaya and Albuquerque (2019)
   - The U.S. radio industry: Jeziorski (2022)
   - TV viewership and advertising: Deng and Mela (2018)
   - Supermarkets in U.K.: Schiraldi, Smith, and Takahashi (2012)
   - Baseball tickets: Lee, Roberts, and Sweeting (2012)
   - Night life in Chicago: Cosman (2017)
   - The U.S. airline industry: Mazur (2017)

Replication Code

- This paper:
  - Implemented in Modern Fortran with OpenMP.
  - Low-level sparse matrix implementations for large state spaces.
  - Simpler solution methods and numerical gradients.
  - `https://github.com/jrblevin/ctgames-qe`
- Other working paper:
  - Python with NumPy/SciPy and Cython.
  - Pre-packaged sparse matrix algorithms from SciPy.
  - More efficient solution methods with analytical derivatives.
  - `https://github.com/jrblevin/ctcomp`

Model and Basic Assumptions

- Infinite horizon game, time indexed by $t \in [0, \infty)$.
- Firms $i = 1, \ldots, N$ maximize expected discounted profits.
- Finite state space $\mathcal{X} \subset \mathbb{R}^L$ with $K = |\mathcal{X}| < \infty$.
- Encoded state space $\mathcal{K} = \{1, \ldots, K\}$.
- Exogenous state changes occur according to $Q_0 = (q_{kl})$.
- Decision times occur at rate λ_{ik}.
- Choice sets $\mathcal{J} = \{0, 1, 2, \ldots, J-1\}$.
- Endogenous state changes induced by actions of players.
- Conditional choice probabilities σ_{ijk}.
- Imply hazards $h_{ijk} = \lambda_{ik}\sigma_{ijk}$.
- Dynamics characterized by a Markov jump process (CTMC).

2×2 Entry Example

- Two firms $i \in \{1, 2\}$.
- Two actions $j \in \{0, 1\}$:
  - $j = 0$: continuation (remain active if active, inactive if inactive)
  - $j = 1$: switching action (enter if inactive, exit if active)
- Two demand states $d \in \{\mathrm{L}, \mathrm{H}\}$.
- State space:

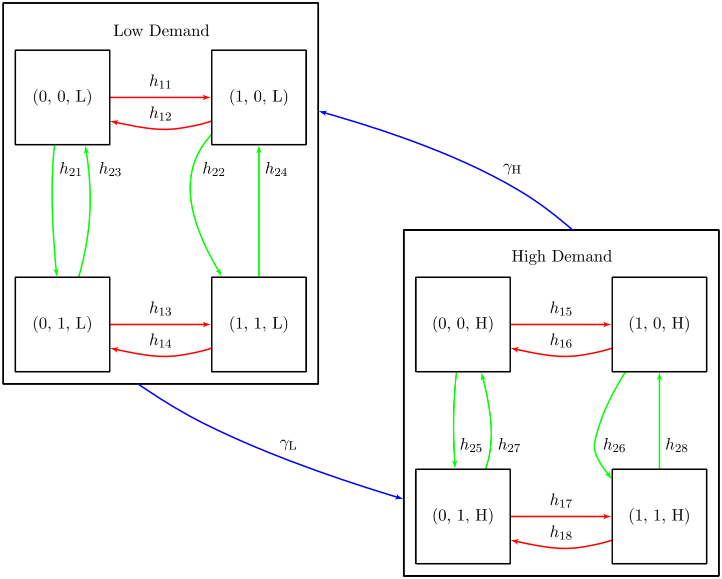
$$\mathcal{X} = \{ \begin{array}{llll} (0,0,\mathrm{L}), & (1,0,\mathrm{L}), & (0,1,\mathrm{L}), & (1,1,\mathrm{L}), \\ (0,0,\mathrm{H}), & (1,0,\mathrm{H}), & (0,1,\mathrm{H}), & (1,1,\mathrm{H}) \end{array} \}$$

- State space in "encoded" form:

$$\mathcal{K} = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

- Let $h_{ik} \equiv h_{i1k}$ denote the hazard of firm i *switching* in state k.
- Let γ_{L} and γ_{H} be the hazards of switching demand states.

2×2 Entry: Figure



2×2 Entry: Intensity Matrix $Q = Q_0 + Q_1 + Q_2$

$$Q = \left[\begin{array}{cccc|cccc} \cdot & h_{11} & h_{21} & 0 & \gamma_L & 0 & 0 & 0 \\ h_{12} & \cdot & 0 & h_{22} & 0 & \gamma_L & 0 & 0 \\ h_{23} & 0 & \cdot & h_{13} & 0 & 0 & \gamma_L & 0 \\ 0 & h_{24} & h_{14} & \cdot & 0 & 0 & 0 & \gamma_L \\ \hline \gamma_H & 0 & 0 & 0 & \cdot & h_{15} & h_{25} & 0 \\ 0 & \gamma_H & 0 & 0 & h_{16} & \cdot & 0 & h_{26} \\ 0 & 0 & \gamma_H & 0 & h_{27} & 0 & \cdot & h_{17} \\ 0 & 0 & 0 & \gamma_H & 0 & h_{28} & h_{18} & \cdot \end{array}\right]$$

Intuition for identification:

- Can determine Q_0, Q_1, and Q_2 since locations of nonzero elements in Q are distinct.
- Instantaneous model is *sparse* even though DT counterpart is *dense*.
- Admissible Q matrices must have the same structure.

Payoffs & Decisions

In between decisions:

- Game remains in some state k.
- Players receive flow payoffs u_{ik}.
- Present discounted payoff in state k over interval $[0, \tau)$:

$$\int_0^\tau \mathrm{e}^{-\rho_i t} \, u_{ik} \, dt.$$

At a decision time:

- Player i chooses action $j \in \{0, \ldots, J-1\}$.
- Player i receives an instantaneous payoff $c_{ijk}(t)$.
- Game moves to a new state $k' = l(i, j, k) \in \mathcal{K}$.

Nature ($i = 0$) changes the state from k to l at rate q_{kl}.

Assumptions

- Bounded rates: for all $i = 1, \ldots, N$, $k, l = 1, \ldots, K$,

$$\rho_i, \lambda_{ik} \in (0, \infty), \quad q_{kl} \in [0, \infty).$$

- Discount rates ρ_i are known.
- Additive separability of instantaneous payoffs at decision times:

$$c_{ijk} = \psi_{ijk} + \varepsilon_{ijk}.$$

- Known error distribution:
  a ε_{ijk} i.i.d. over decision times, players, and states;
  b F absolutely continuous with respect to Lebesgue measure;
  c $\mathrm{E}[\varepsilon_{ijk}] < \infty$;
  d support of ε_{ijk} is $\mathbb{R}$.
- Convenient sufficient condition: ε_{ijk} are iid TIEV.

Rust (1987) Example

State variable is accumulated mileage

$$\mathcal{K} = \{1, \ldots, K\}$$

with $K \times K$ intensity matrix

$$Q_0 = \begin{bmatrix} -\gamma & \gamma & 0 & 0 & \cdots & 0 \\ 0 & -\gamma & \gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma & \gamma & 0 \\ 0 & 0 & \cdots & 0 & -\gamma & \gamma \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Multi-state jumps over a time *interval* are still possible.

Rust (1987) Example

At rate λ_k, Harold Zurcher decides whether or not to replace a bus engine in mileage state k: $\mathcal{J} = \{0, 1\}$.

Cost minimization problem:

- flow utility u_k received while in state k,
- continuation ($j = 0$) is costless with $\psi_{0k} = 0$,
- replacement cost $\psi_{1k} = \mu < 0$ paid upon replacement ($j = 1$),
- plus i.i.d. shocks ε_{jk} in each case.

Solving the dynamic program yields probability of replacement: σ_{1k}

Rust (1987) Example

Intensity matrix for the agent:

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \lambda_2 \sigma_{12} & -\lambda_2 \sigma_{12} & 0 & 0 & \cdots & 0 \\ \lambda_3 \sigma_{13} & 0 & -\lambda_3 \sigma_{13} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1} \sigma_{1,K-1} & 0 & \cdots & 0 & -\lambda_{K-1} \sigma_{1,K-1} & 0 \\ \lambda_K \sigma_{1K} & 0 & \cdots & 0 & 0 & -\lambda_K \sigma_{1K} \end{bmatrix}.$$

Aggregate intensity matrix: $Q = Q_0 + Q_1$.

Value Functions with N Players

Let ς_i denote player i's beliefs about rival choice probabilities. The value function for player i in state k is

$$V_{ik}(\varsigma_i) = \frac{1}{\rho_i + \sum_{l \neq k} q_{kl} + \sum_m \lambda_{mk}} \times \Bigg[u_{ik} + \sum_{l \neq k} q_{kl} V_{il}(\varsigma_i)$$

$$+ \sum_{m \neq i} \lambda_{mk} \sum_{j=0}^{J-1} \varsigma_{imjk} V_{i,l(m,j,k)}(\varsigma_i)$$

$$+ \lambda_{ik} \, \mathrm{E} \max_j \left\{ \psi_{ijk} + \varepsilon_{ijk} + V_{i,l(i,j,k)}(\varsigma_i) \right\} \Bigg].$$

Markov Perfect Equilibrium

Following Maskin and Tirole (2001) and empirical discrete-time games literature: Markov perfect equilibria in pure strategies.

Definition

A *Markov perfect equilibrium* (MPE) is a collection of stationary Markov policy rules $\{\delta_i^*\}_{i=1}^N$ such that for each player i and for all (k, ε_{ik}):

$$\delta_i^*(k, \varepsilon_{ik}) = \arg\max_j \left\{\psi_{ijk} + \varepsilon_{ijk} + V_{i,l(i,j,k)}(\varsigma_i)\right\} \quad \text{(best response)}$$

$$\varsigma_{imjk} = \Pr\left[\delta_m^*(k, \varepsilon_{mk}) = j \mid k\right] \text{ for all } m \neq i \quad \text{(consistent beliefs)}$$

Equilibrium CCPs: $\sigma_{ijk} = \Pr\left[\delta_i^*(k, \varepsilon_{ik}) = j \mid k\right]$

Theorem 1: Existence of MPE

Theorem

Under the maintained assumptions, a Markov perfect equilibrium exists.

Proof is straightforward:

- Define $\Upsilon : [0,1]^{N \times J \times K} \rightarrow [0,1]^{N \times J \times K}$ by stacking best response probabilities.
- Υ is continuous on compact set $[0,1]^{N \times J \times K}$.
- By Brouwer's fixed point theorem, Υ has a fixed point.
- Fixed point probabilities imply stationary Markov strategies that constitute an MPE.

Theorem 2: Linear Representation of V_i

- CCP inversion of ABBE yields a linear representation of $V_i(\sigma)$.
- Useful for both identification and estimation.

Theorem

Under the maintained assumptions, for a given collection of equilibrium CCPs σ, V_i has the following linear representation:

$$V_i(\sigma) = \Xi_i^{-1}(\sigma)\left[u_i + L_i C_i(\sigma_i)\right] \tag{1}$$

$$\Xi_i(\sigma) = \rho_i I_K + \sum_{m=1}^{N} L_m[I_K - \Sigma_m(\sigma_m)] - Q_0 \tag{2}$$

where $\Xi_i(\sigma)$ is a nonsingular $K \times K$ matrix, where:

- $L_m = \mathrm{diag}(\lambda_{m1}, \ldots, \lambda_{mK})$*: diagonal matrix of rates,*
- $\Sigma_m(\sigma_m)$*: transition matrix implied by player m's CCPs,*
- $C_i(\sigma_i)$*: expected instantaneous payoffs given player i's CCPs.*

Overview of Identification Strategy

- Implications of structural model:

$$\theta \mapsto \{u_i, \psi_i, \lambda_i, V_i, \sigma_i\} \mapsto \{h_i, Q_i\} \mapsto Q \mapsto P(\Delta).$$

- Identification analysis:

$$P(\Delta) \mapsto Q \mapsto \{Q_i, h_i\} \mapsto \{\lambda_i, \sigma_i, V_i, \psi_i, u_i\} \mapsto \theta$$

- Overview of identification results:
  - Address aliasing problem in $P(\Delta) \mapsto Q$ using nonparametric structural restrictions
  - Identifying restrictions on $\{h_i, V_i, \psi_i\}$ from model structure.
  - Given identified quantities, apply Theorem 2 to identify u_i.

Identification of Q

With equispaced discrete time observations, $P(\Delta) = (P_{kl}(\Delta))$ is observable but Q is not.

Is there a unique matrix Q such that

$$P(\Delta) = \exp(\Delta Q) \equiv \sum_{j=0}^{\infty} \frac{(\Delta Q)^j}{j!} = I + Q + \frac{1}{2} Q^2 + \ldots ?$$

In discrete time with τ subperiods, is there a unique P_0 with

$$P(\Delta) = P_0^{\tau} ?$$

For both questions, the answer is no in general.

Identification of Q: Restrictions on $P(\Delta)$

Sufficient conditions for identification of Q:

- $P(\Delta)$ has distinct, real, and positive eigenvalues.
- Q has distinct, *real* eigenvalues.
- $\min_i\{P_{ii}(\Delta)\} > 1/2$.
- $\det P(\Delta) > e^{-\pi}$.
- Unclear which structures satisfy these "top-down" conditions...

Alternative sampling schemes:

- $\Delta < \overline{\Delta}$.
- $P(\Delta_1)$ and $P(\Delta_2)$ with $\Delta_1 \neq k\Delta_2$, $k \in \mathbb{N}$.
- Different observation intervals help, but often not available...

Phillips (1973): Economic models usually restrict Q itself, rather than $P(\Delta)$.

Identification of Q: Prior Restrictions on Q

Assumption

Q has distinct eigenvalues $\mu_1, \ldots, \mu_K$ that do not differ by an integer multiple of $2\pi i/\Delta$.

By Gantmacher (1959) and Phillips (1973), all solutions $\tilde{Q}$ to $\exp(\Delta\tilde{Q}) = P(\Delta)$ satisfy

$$\tilde{Q} = Q + VDV^{-1}$$

with $Q = V\Lambda V^{-1}$

$$D = \frac{2\pi i}{\Delta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & -M \end{bmatrix}, M = \mathrm{diag}(m_1, \ldots, m_\rho), m_i \in \mathbb{Z}\,.$$

Without restrictions, the following are identified:

- Eigenvectors V,
- Number of complex eigenvalues 2ρ,
- Real eigenvalues $\mu_{2\rho+1}, \ldots, \mu_K$.

Identification of Q: Linear restrictions on Q

Blevins (2017) derived a rank condition under which $\lfloor \frac{K}{2} \rfloor$ linear restrictions on $\operatorname{vec} Q$ are sufficient for a general $K \times K$ matrix Q:

$$R \operatorname{vec}(Q) = r.$$

Any other solutions $\tilde{Q}$ must also satisfy the prior restrictions on Q to be admissible.

Specializing this to the case of *intensity matrices*, we derive conditions for identification of Q using only $\lfloor \frac{K-1}{2} \rfloor$ restrictions, focusing on nonparametric restrictions from the model structure.

Theorem 3: Identification of Q

Theorem

Suppose the state vector is $x = (x_0, x_1, \dots, x_N) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ where the component $x_0 \in \mathcal{X}_0$ is an exogenous market characteristic taking $|\mathcal{X}_0| = K_0$ values and for each $i = 1, \dots, N$ the component x_i is a player-specific state affected only by the action of each player with $|\mathcal{X}_i| = K_1$ possible distinct values. If Q has distinct eigenvalues that do not differ by an integer multiple of $2\pi i/\Delta$, then Q is identified when

$$K_0 K_1^N - K_0 - NJ + \frac{1}{2} \geq 0. \tag{3}$$

The quantity on the left is strictly increasing in K_1, strictly increasing in K_0 when $K_1 > 1$, and strictly decreasing in J.

Intuition for the Order Condition

- State vector is $x = (x_0, x_1, \ldots, x_N)$ where
  - x_0 is a common state taking K_0 values (exogenous),
  - x_i is a firm-i-specific state taking K_1 values,
  - x_i only affected by the actions of player i.
- Each player i has J choices. Total states: $K = K_0 K_1^N$.
- Need $\lfloor (K-1)/2 \rfloor$ linear restrictions on Q.
- We have $K - J(J-1) - (K_0 - 1) - 1$ known zeros per row!
- Order condition satisfied when: $K_0 K_1^N - K_0 - NJ + \frac{1}{2} \geq 0$.

Examples where Q is identified:

- The 2×2 entry model
- Single-agent renewal model when $K \geq 3$
- All nontrivial ($K_1 \geq 2$) binary choice games
- All three-choice games with $K_1 \geq 3$

Identification of the Structural Primitives

- With Q in hand, we turn to the structural primitives.
- Note that $h_{ijk} = \lambda_{ik}\sigma_{ijk}$ identified for $j > 0$.
- With T1EV errors, hazard analog of CCP inversion:

$$\ln h_{ijk} = \ln h_{i0k} + \psi_{ijk} + V_{i,l(i,j,k)} - V_{ik}.$$

- Stacking across states and actions gives a linear system:

$$\begin{bmatrix} \ln h_{i1} \\ \vdots \\ \ln h_{i,J-1} \end{bmatrix} = \left[\begin{array}{c|ccccc|c} I_K & I_K & 0 & \dots & 0 & S_{i1} - I_K \\ I_K & 0 & I_K & \dots & 0 & S_{i2} - I_K \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_K & 0 & 0 & \dots & I_K & S_{i,J-1} - I_K \end{array}\right] \left[\begin{array}{c} \ln h_{i0} \\ \hline \psi_{i1} \\ \vdots \\ \psi_{i,J-1} \\ \hline V_i \end{array}\right].$$

- $S_{i,j}$ = transition matrix induced by firm i making choice j.

Theorem 4: Identification of the Structural Primitives

Theorem

For each player i, the augmented system with linear restrictions is:

$$\begin{bmatrix} \ln h_i^{+} \\ r_i \end{bmatrix} = \begin{bmatrix} X_i \\ R_i \end{bmatrix} \begin{bmatrix} \ln h_i^{0} \\ \psi_i \\ V_i \end{bmatrix},$$

where X_i is an identified $(J-1)K \times (J+1)K$ matrix with rank $(J-1)K$. If R contains $2K$ additional full-rank restrictions such that $\begin{bmatrix} X_i \\ R_i \end{bmatrix}$ has rank $(J+1)K$, then h_i^0, ψ_i, and V_i are identified.

In CT, number of restrictions is linear in N while in DT it is exponential in N (Pesendorfer and Schmidt-Dengler, 2008).

Finding Restrictions for Identification

- Constant move arrival rates: $\lambda_{ik} = \lambda_i$ gives $K - 1$ restrictions
- Constant instantaneous payoffs: $\psi_{ijk} = \psi_{ij}$ gives $(J - 1)(K - 1)$ restrictions
- Exclusion restrictions: $V_{ik} = V_{ik'}$ when states are payoff-equivalent
- Terminal states: $V_{ik} = 0$ in absorbing states

Example: Binary choice ($J = 2$) with $\lambda_{ik} = \lambda_i$ and $\psi_{i1k} = \psi_{i1}$:
$(K - 1) + (K - 1) = 2K - 2$ restrictions $\Rightarrow$ need only 2 more

- Not including cross-player & shape restrictions.
- Rank condition can usually be verified by inspection.

Theorem 5: Identification of the Flow Payoffs u_i

Theorem

Suppose the above assumptions hold. If for any player i the quantities V_i, ψ_i, and Q are identified, then the flow payoffs u_i are also identified.

The proof follows from using the linear representation from Theorem 2, noting that all quantities other than u_i are identified, and solving for u_i.

Estimation with Discrete Time Data

- Markets $m = 1, \ldots, M$ are independent.
- Sample for market m consists of states $\{k_{mt}\}$ observed at times Δt for $t = 1, \ldots, T$.
- We cannot see the actual sequence of events.
- Observations are sampled at regular intervals of length Δ.
- Estimate θ using implied transition matrix $P_\theta(\Delta)$.
- Log-likelihood for a sample $\{\{k_{mt}\}_{t=1}^{T}\}_{m=1}^{M}$:

$$\ln L_M(\theta) = \sum_{m=1}^{M} \sum_{t=1}^{T} \ln[P_\theta(\Delta)]_{k_{m,t-1},k_{mt}}.$$

Single Agent Monte Carlo Experiments

- Based on bus engine replacement model (Rust, 1987).
- State space (mileage): $\mathcal{X} = \{1, \dots, 90\}$.
- Parameters are cost of mileage β, replacement cost μ, rate of mileage increase γ, and decision rates λ_{ik}.
- Buses $m = 1, \dots, M$ observed for T_m months.
- First, we use the real data to estimate parameters.
- Three specifications for λ_{ik}:

$$\lambda_{ik} = 1, \qquad \lambda_{ik} = \lambda, \qquad \lambda_{ik} = \begin{cases} \lambda_{\text{L}} & \text{if } k \leq \left\lfloor \frac{K}{2} \right\rfloor, \\ \lambda_{\text{H}} & \text{otherwise.} \end{cases}$$

- CT discount factor $\rho = 0.05$ (DT $\beta = \mathrm{e}^{-0.05} \approx 0.95$).
- Sample contains $M = 162$ buses with $T_m \in \{24, \dots, 125\}$.
- Total of 15,402 discrete-time bus-month observations.

Estimates with Rust (1987) Data

	Fixed $\lambda = 1$		Variable λ		Het. λ	
	Est.	S.E.	Est.	S.E.	Est.	S.E.
Dec. Rate (λ)	1.000	–	0.032	(0.005)	–	–
Dec. Rate 1 (λ_{L})	–	–	–	–	0.022	(0.004)
Dec. Rate 2 (λ_{H})	–	–	–	–	0.033	(0.005)
Mil. Rate (γ)	0.526	(0.006)	0.526	(0.006)	0.526	(0.006)
Mil. Cost (β)	-0.533	(0.052)	-1.257	(0.285)	-1.711	(0.493)
Repl. Cost (μ)	-8.081	(0.393)	-8.072	(1.345)	-9.643	(2.189)
LL	-13947.55		-13938.51		-13937.66	
Obs.	15406		15406		15406	
Test for $H_0 : \lambda_{\mathrm{L}} = \lambda_{\mathrm{H}} = 1$						
LR	–		18.08		19.78	
p-value	–		0.00002		0.00005	
Test for $H_0 : \lambda_{\mathrm{L}} = \lambda_{\mathrm{H}}$						
LR	–		–		1.70	
p-value	–		–		0.1923	

Conclusions from Rust (1987) Data

- Estimated decision rates are quite different from 1.
- We strongly reject $\lambda = 1$, but fail to reject $\lambda_{\mathsf{L}} = \lambda_{\mathsf{H}}$.
- Relatively low rate of monitoring, but a higher cost of mileage.
- With forced monthly decisions: model compensates w/lower mileage cost to fit observed replacement timing.

Single Agent Monte Carlo Experiments

- Choose the Monte Carlo parameters based on estimates:

$$(\lambda_{\mathsf{L}}, \lambda_{\mathsf{H}}, \gamma, \beta, \mu) = (0.05, 0.10, 0.5, -2.0, -9.0).$$

- This allows us to interpret 1 unit of time as 1 month.
- We simulate data over $t \in [0, 120]$ (10 years) for M markets.
- We vary M from 200 to 3200.
- CT data and DT data with $\Delta \in \{0.0, 1.0, 8.0\}$.
- Report mean and s.d. over 100 replications.

Single Agent Monte Carlo Results

M	Sampling		λ_L	λ_H	γ	β	μ
∞	DGP	True	0.050	0.100	0.500	-2.000	-9.000
200	Continuous	Mean	0.050	0.100	0.500	-2.050	-9.178
		S.D.	0.007	0.008	0.004	0.310	1.096
200	$\Delta = 1.00$	Mean	0.051	0.100	0.508	-2.079	-9.235
		S.D.	0.007	0.008	0.004	0.317	1.117
200	$\Delta = 8.00$	Mean	0.051	0.100	0.508	-2.093	-9.284
		S.D.	0.009	0.009	0.005	0.374	1.281
800	Continuous	Mean	0.050	0.100	0.500	-1.988	-8.957
		S.D.	0.003	0.005	0.002	0.121	0.427
800	$\Delta = 1.00$	Mean	0.051	0.101	0.508	-2.011	-8.999
		S.D.	0.003	0.005	0.002	0.124	0.433
800	$\Delta = 8.00$	Mean	0.051	0.100	0.508	-2.018	-9.020
		S.D.	0.003	0.005	0.003	0.145	0.498
3200	Continuous	Mean	0.050	0.100	0.500	-1.995	-8.999
		S.D.	0.002	0.002	0.001	0.072	0.238
3200	$\Delta = 1.00$	Mean	0.051	0.100	0.508	-2.014	-9.025
		S.D.	0.002	0.002	0.001	0.072	0.233
3200	$\Delta = 8.00$	Mean	0.051	0.100	0.508	-2.009	-9.004
		S.D.	0.002	0.002	0.001	0.075	0.244

Quality Ladder Model

Following Ericson and Pakes (1995), Pakes and McGuire (1994):

- N firms with products of quality $\omega_i \in \{1, 2, \dots, \bar{\omega}, \bar{\omega} + 1\}$
  - States 1 to $\bar{\omega}$: active incumbent firms
  - State $\bar{\omega} + 1$: inactive/potential entrants
- Heterogeneous move arrival rates:
  - λ_{H}: high quality firms ($\omega_i \geq \omega^{\mathrm{h}}$)
  - λ_{L}: low quality firms ($\omega_i < \omega^{\mathrm{h}}$) and potential entrants
- Firm decisions:
  - Incumbents: continue, invest κ to upgrade quality, or exit (scrap value ϕ)
  - Potential entrants: enter at cost η w/quality ω^{e}, or stay out
- Flow costs/profits: fixed cost μ, profits π_{ik} from Nash-Bertrand competition, logit demand model.
- Industry-wide negative shocks at rate γ (outside alternative improvement)

Quality Ladder Model: Monte Carlo Setup

- Model specifications:
  - Number of firms: $N = 2$ to 30
  - Quality levels: $\bar{\omega} = 7$, entry at $\omega^e = 4$, threshold $\omega^h = 4$.
  - State space size: K ranges from 56 to 58+ million states
- Simulation details:
  - Time horizon: $T = 120$ (CT and DT with $\Delta = 1$)
  - Market size $\bar{M}$ increases with N to maintain reasonable n_{avg}
  - 100 replications per specification

Quality Ladder Model: Computational Time

N	$\bar{\omega}$	K	$\bar{M}$	Obtain V
2	7	56	0.40	0.15 sec.
4	7	840	0.60	0.27 sec.
6	7	5,544	0.75	0.65 sec.
8	7	24,024	0.85	3 sec.
10	7	80,080	0.95	10 sec.
12	7	222,768	1.05	30 sec.
14	**7**	**542,640**	**1.15**	**1.3 min.**
16	7	1,193,808	1.20	3.3 min.
18	7	2,422,728	1.25	7.0 min.
20	7	4,604,600	1.30	13.7 min.
22	7	8,288,280	1.35	27.5 min.
24	7	14,250,600	1.40	49.4 min.
26	7	23,560,992	1.45	1.8 hr.
28	7	37,657,312	1.50	3.0 hr.
30	7	58,433,760	1.55	4.9 hr.

Doraszelski and Judd (2012): $N = 14$, approx. one year in DT.

Quality Ladder Model: Monte Carlo

N	K	Sampling		λ_L	λ_H	γ	κ	η	μ
		DGP	True	1.000	1.200	0.400	0.800	4.000	0.900
2	56	Continuous	Mean	0.997	1.196	0.400	0.796	3.988	0.899
			S.D.	0.015	0.020	0.010	0.032	0.137	0.021
		$\Delta = 1.0$	Mean	1.021	1.223	0.399	0.801	3.932	0.914
			S.D.	0.177	0.181	0.007	0.283	0.841	0.063
4	840	Continuous	Mean	0.999	1.198	0.397	0.806	4.030	0.897
			S.D.	0.013	0.018	0.014	0.033	0.160	0.022
		$\Delta = 1.0$	Mean	0.998	1.197	0.400	0.781	3.948	0.902
			S.D.	0.114	0.113	0.006	0.180	0.456	0.040
6	5,544	Continuous	Mean	1.001	1.198	0.399	0.798	4.013	0.900
			S.D.	0.014	0.018	0.016	0.035	0.144	0.021
		$\Delta = 1.0$	Mean	1.004	1.207	0.399	0.805	4.017	0.901
			S.D.	0.087	0.088	0.006	0.135	0.330	0.032
8	24,024	Continuous	Mean	1.000	1.200	0.400	0.802	4.027	0.899
			S.D.	0.013	0.017	0.018	0.033	0.149	0.023
		$\Delta = 1.0$	Mean	1.012	1.213	0.400	0.814	4.030	0.905
			S.D.	0.082	0.083	0.005	0.125	0.292	0.030

Conclusion

- Identification of move arrival rates in the ABBE model.
- Theoretical properties:
  - Existence of Markov perfect equilibrium.
  - Linear representation of value function in terms of CCPs.
- Econometric properties:
  - Identification of Q, λ, σ, V, ψ, and u.
  - Degree of underidentification less severe than in DT.