Nonparametric Identification of Dynamic Decision Processes with Discrete and Continuous Choices*

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Abstract. This paper establishes conditions for nonparametric identification of dynamic optimization models in which agents make both discrete and continuous choices. We consider identification of both the payoff function and the distribution of unobservables. Models of this kind are prevalent in applied microeconomics and many of the required conditions are standard assumptions currently used in empirical work. We focus on conditions on the model that can be implied by economic theory and assumptions about the data generating process that are likely to be satisfied in a typical application. Our analysis is intended to highlight the identifying power of each assumption individually, where possible, and our proofs are constructive in nature.

Keywords: nonparametric identification, Markov decision processes, dynamic decision processes, discrete choice, continuous choice.

JEL Classification: C14, C25, C23, C51.

1. Introduction

This paper establishes conditions for nonparametric identification of both the payoff function and the distribution of unobservables in dynamic decision processes in which agents make both discrete and continuous choices. We focus on finding conditions on the model that can be implied by economic theory and conditions on the data generating process which are likely to be satisfied in a typical application.

Models of this kind are routinely estimated in applied microeconomics, particularly in empirical studies in industrial organization and labor economics. Examples include

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dynamic investment models with discrete entry and exit decisions and continuous investment choices and models with discrete labor force participation decisions and a continuous choice over hours of work. See Ackerberg, Benkard, Berry, and Pakes (2007) and Keane, Todd, and Wolpin (2011) for surveys of related models in industrial organization and labor economics, respectively.

Although our proofs are constructive in nature, this paper does not consider nonparametric estimation of the model, which is often prohibitive in applications due to both data limitations and the curse of dimensionality. Still, formally establishing conditions for nonparametric identification allows us to better understand the identifying power of various model assumptions and sources of variation in the data and provides a benchmark for practitioners when specifying and estimating parametric models. Finally, since the period utility function is the nonparametric object of interest, the results herein may be useful as a basis for welfare analysis or for considering identification of counterfactual model predictions.

There is an extensive body of work on identification and estimation of dynamic discrete choice models which has largely focused on models with discrete states and where the distribution of unobservables is known, following the framework of Rust (1994). Magnac and Thesmar (2002) provided several nonparametric identification results under the assumption that the distribution of unobservables is known, but without further restricting the payoffs. The nested fixed point estimator of Rust (1987), conditional choice probability estimator of Hotz and Miller (1993), and the nested pseudo likelihood estimator of Aguirregabiria and Mira (2002) have all been widely used in applied work. In a related model, Jofre-Bonet and Pesendorfer (2003) considered nonparametric identification of the cost distribution in a dynamic auction game with continuous choices. Heckman and Navarro (2007) and Aguirregabiria (2010) studied identification of dynamic discrete choice models with continuous state variables. Hong and Shum (2010) considered a dynamic continuous choice model with continuous states and parametric payoffs.

Our results on nonparametric identification of the distribution of unobservables extend a long line of research on semiparametric and nonparametric identification of static discrete choice and continuous outcome models including Manski (1975, 1985), Matzkin (1991, 1992, 1993, 2003), Dubin and McFadden (1984), and Newey (2007). Our conditions are most closely related to Matzkin (2003), who used monotonicity to nonparametrically identify nonlinear, nonseparable models, and Matzkin (1993), who used shape restrictions on the subutility functions to nonparametrically identify static discrete choice models.

Our work is also related to the literature on identification and estimation of dynamic discrete choice games (Aguirregabiria and Mira, 2007; Pesendorfer and Schmidt-Dengler, 2007; Pakes, Ostrovsky, and Berry, 2007; Bajari, Benkard, and Levin, 2007; Bajari, Chernozhukov, Hong, and Nekipelov, 2007; Srisuma and Linton, 2012) and continuous or monotone choice games (Shrimpf, 2011; Srisuma, 2013). See Aguirregabiria and Mira (2010) for a recent survey. Although our focus is on single-agent models, rather than games, our results should be a useful starting point for studying identification of games with both discrete and continuous choices.[1]

The contributions of this paper are to establish conditions for the nonparametric identification of models in which agents make a dynamic continuous choice in addition to the usual discrete choice. Existing work on identification of pure discrete choice models provides a useful starting point for analyzing the models we consider, but given the continuous choice additional consideration is needed to find conditions for identifying the primitives of interest, which are infinite-dimensional functions rather than finite-dimensional vectors. Thus, even though the observable distributions are richer in this setting it is unclear whether such models are nonparametrically identified because the unknown primitives are themselves more complex.

This paper proceeds as follows. Section 2 introduces the model and basic assumptions. We then work towards our nonparametric identification results in two steps. First, we suppose that the distribution of unobservables is known or identified and consider semi-nonparametric identification of the policy and payoff functions in Section 3. Then, we consider nonparametric identification of the distribution of unobservables in Section 4. This sequential approach allows us to highlight the identifying power of each assumption individually. If the corresponding conditions from both sections are satisfied, then the model is nonparametrically identified. Finally, Section 5 concludes.

2. Framework and Basic Assumptions

Consider a discrete-time dynamic optimization model with an infinite time horizon indexed by $t = 1, 2, \ldots, \infty$. Suppose the state of the market at time t that is observable by the researcher can be summarized by random vector $S_t \in \mathcal{S} \subseteq \mathbb{R}^L$. Additionally, let $\varepsilon_t \in \mathcal{E} \subseteq \mathbb{R}^{K+1}$ and $\eta_t \in \mathcal{H} \subseteq \mathbb{R}$ denote the remaining state variables that are not observable by the researcher but affect, respectively, the discrete and continuous choices made by the agent. The joint state $(S_t, \varepsilon_t, \eta_t)$ evolves according to a controlled, time-homogeneous, first-order Markov process which we describe below.

The timing of actions and information revelation within each period is as follows:

1. At the beginning of the period, the agent observes the realization of S_t and the random vector ε_t. Given S_t and ε_t, the agent makes a discrete choice $D_t \in \mathcal{D} =$

[1] A previous version of this paper includes results for dynamic games with discrete and continuous choices under a completeness assumption and an exclusion restriction (Blevins, 2010).

$\{0, 1, \ldots, K\}$.

2. After making the discrete choice D_t, the agent observes the realization of the random variable η_t. Given S_t, ε_t, D_t, and η_t, the agent makes a continuous choice $C_t \in \mathcal{C} \subseteq \mathbb{R}$ and receives a payoff $U(D_t, C_t, S_t, \varepsilon_t, \eta_t)$.

The state S_t and the choices D_t and C_t are observed by the researcher, but ε_t and η_t are only observed by the agent. The agent is forward-looking and discounts future payoffs using the discount factor β. The agent has rational expectations about the evolution of S_t, ε_t, and η_t and chooses the actions D_t and C_t in each period in order to maximize expected discounted future utility.

Regarding the sequential nature of decisions within a period, we assume that the distribution of η_t is in the agent's information set when the discrete choice D_t is made, although the realization of η_t is not yet known. Therefore, the agent can make forecasts and evaluate expectations involving η_t. In particular, agents choose D_t to maximize the expected present discounted value of payoffs.

We make the following additional assumptions, most of which are standard assumptions used in the dynamic discrete choice literature (cf. Rust, 1994; Aguirregabiria and Mira, 2010), with slight modifications in some cases to allow for the additional continuous choice. We discuss each assumption in detail below. The relationships between the state variables and decision variables for two full periods are also summarized graphically in Figure 1. Direct relationships that are prohibited by the assumptions are indicated by dotted lines.

Assumption 1 (Additive Separability)**.** Let $\mathcal{U} = \mathcal{D} \times \mathcal{S} \times \mathcal{C} \times \mathcal{H}$. There exists a function $u : \mathcal{U} \to \mathbb{R}$ such that for all $(d_t, c_t, s_t, \eta_t) \in \mathcal{U}$ and all $\varepsilon_t \in \mathcal{E}$,

$$U(d_t, c_t, s_t, \varepsilon_t, \eta_t) = u(d_t, c_t, s_t, \eta_t) + \varepsilon_{d_t,t}.$$

Assumption 2 (Conditional Independence)**.** For all $s_t \in \mathcal{S}$, $e_t \in \mathcal{E}$, and $h_t \in \mathcal{H}$, the conditional cdf of the controlled Markov process can be factored as follows, almost surely:

$$F_{S_t,\varepsilon_t,\eta_t|S_{t-1},\varepsilon_{t-1},\eta_{t-1},D_{t-1},C_{t-1}}(s_t, e_t, h_t) = F_{S_t|S_{t-1},D_{t-1},C_{t-1}}(s_t) \cdot F_{\varepsilon_t,\eta_t|S_t=s_t}(e_t, h_t).$$

Assumption 3 (Continuous Choice Unobservables)**.** For all t and $h \in \mathcal{H}$, $F_{\eta_t|D_t,S_t,\varepsilon_t}(h) = F_{\eta_t|D_t,S_t}(h)$ almost surely.

Assumption 4 (Discrete Choice Unobservables)**.** The distribution of $\varepsilon_t = (\varepsilon_{0,t}, \ldots, \varepsilon_{K,t})$ has support $\mathcal{E} = \mathbb{R}^{K+1}$ and $\mathrm{E}[\varepsilon_{d,t}] < \infty$ for all $d \in \mathcal{D}$ and all t.

Both Assumptions 1 and 2 have been widely used in dynamic discrete choice models since Rust (1987, 1988) demonstrated their role in generating empirically tractable

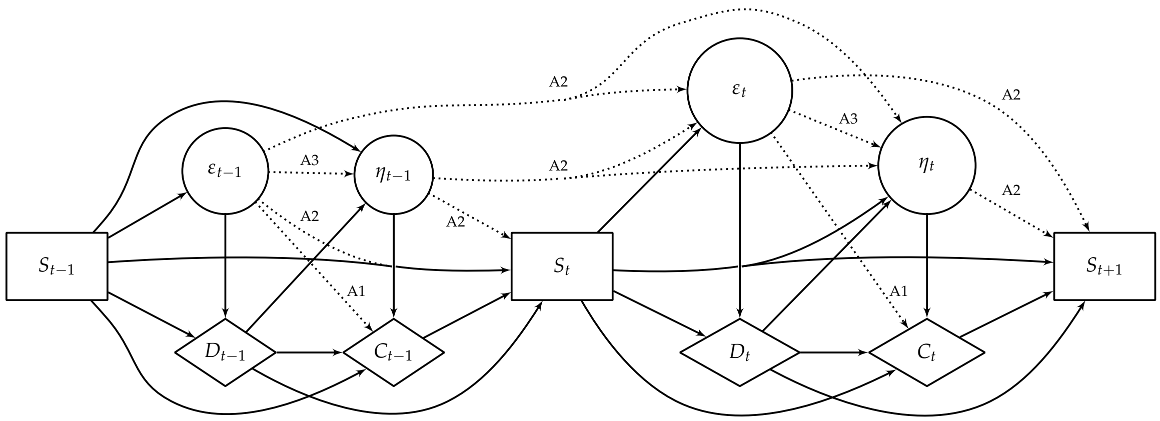

FIGURE 1. State Variable Evolution

structural models of dynamic discrete choice (cf. Rust, 1994; Aguirregabiria and Mira, 2010). Assumption 1 is an additive separability condition of the sort used in both static and dynamic discrete choice analysis (cf. McFadden, 1974; Rust, 1994). Note that in our context, although it requires ε_t to affect payoffs additively, η_t may still affect payoffs in a non-separable manner. Assumption 2 is the conditional independence assumption of Rust (1987, 1988, 1994), which limits the serial dependence of the unobservables.[2] Under this assumption, the random variables ε_t and η_t are independent of ε_{t-1} and η_{t-1} conditional on S_t. Furthermore, S_t is independent of ε_{t-1} and η_{t-1} conditional on S_{t-1}, D_{t-1}, and C_{t-1}.

In light of the continuous choice, we employ Assumption 3 in order to maintain the same additive separability in ε_t for the dynamic payoffs that Assumptions 1 and 2 yield for pure dynamic discrete choice models.[3] Under this assumption, η_t, the unobservable associated with the continuous choice, may depend on the discrete choice, D_t, and the state, S_t, but not directly on the vector of discrete-choice-specific unobservables, ε_t. Importantly though, D_t is itself determined by S_t and ε_t, so this allows for selection effects and heterogeneity in the distribution of η_t following the discrete choice. Finally, Assumption 4 is a regularity condition on the distribution of the discrete-choice-specific unobservables. Given the Markov assumption and the time invariance of both the period utility function and state transition kernel, the agent's optimal decision rules are stationary. As such, we will omit the time subscript from variables when the context is clear.

[2]In light of recent work by Hu and Shum (2012), our results can also be applied to models with serially correlated unobserved state variables. Under their assumptions, the joint Markov kernel of the observed and unobserved states is nonparametrically identified and can be treated as known for our subsequent analysis.

[3]Maintaining this structure is important for preserving desirable computational properties, such as the possibility of using well-known integration methods such as GHK (cf. Hajivassiliou and Ruud, 1994) and widely used closed forms for choice probabilities under the type I extreme value distribution.

The timing of the model is such that the discrete choice D is made at the beginning of the period, prior to learning the value of η and prior to making the continuous choice C. Under the assumptions above, the value function from the perspective of the beginning of the period can be expressed recursively as

$$(1) \qquad V(s,\varepsilon) = \max_{d\in\mathcal{D}} \{v(d,s) + \varepsilon_d\},$$

where $v(d,s)$ is defined as

$$(2) \qquad v(d,s) \equiv \mathrm{E}\left[\max_{c\in\mathcal{C}} \left\{u(d,c,s,\eta) + \beta\,\mathrm{E}[V(s',\varepsilon') \mid d,c,s]\right\} \,\middle|\, d,s\right].$$

We call v the *discrete-choice-specific value function*. This function gives the expected discounted utility obtained by choosing alternative d when the current state is s, net of ε_d, assuming continued optimal behavior in future periods. This function is similar to the choice-specific value function in dynamic discrete choice models (Rust, 1994), but $v(d,s)$ here represents an expected valuation at an intermediate point in the period, before the continuous-choice-specific shock is known.

As can be seen in (1) and (2), Assumptions 1–3 allow us to express this problem in an analytically and empirically tractable form that resembles a static discrete choice problem, with the discrete-choice-specific value function playing the role of the period utility function. First, Assumption 1 allowed us to write (1). Then, Assumption 2 allowed us to omit ε and η from the conditioning set in the inner expectation over s' and ε' in (2). Similarly, Assumption 3 allowed us to omit ε from the conditioning set of the outer expectation over η. Therefore, as we alluded to earlier, Assumption 3 is needed here to ensure that the discrete choice problem is effectively additively separable in ε, as is typically assumed in pure dynamic discrete choice models. Otherwise, if η could be arbitrarily correlated with ε, the discrete-choice-specific value function could depend on ε in a non-separable manner even under the additive separability condition in Assumption 1.

We will also make use of the *ex-ante value function*:

$$\bar{V}(s) \equiv \mathrm{E}\left[V(s,\varepsilon) \mid s\right] = \mathrm{E}\left[\max_{d\in\mathcal{D}} \{v(d,s) + \varepsilon_d\} \,\middle|\, s\right].$$

Using this function and the law of iterated expectations, we can rewrite (2) as

$$(3) \qquad v(d,s) = \mathrm{E}\left[\max_{c\in\mathcal{C}} \left\{u(d,c,s,\eta) + \beta\,\mathrm{E}\left[\bar{V}(s') \mid d,c,s\right]\right\} \,\middle|\, d,s\right].$$

The structural primitive of interest here is the payoff function u.

Example 1. Timmins (2002) considers the problem of a municipal water utility administrator who must choose the price of water each period. The marginal price may either be zero, or it may be some positive value. Consider a stylized version of the model in which $D \in \mathcal{D} = \{0, 1\}$ represents the decision of whether to use flat rate pricing or metered, per unit pricing. Then, conditional on a pricing regime, $C \in \mathcal{C} = [0, \infty)$ represents the choice of the price. If $D = 0$, then C is the flat rate price (e.g., a monthly service charge). If $D = 1$, then C is the marginal price (e.g., price per hundred cubic feet). These decisions are functions of the demand for water, the groundwater stock, extraction costs, etc.

Associated with each discrete choice $d \in \mathcal{D}$ is a random variable ε_d which captures costs associated with each pricing regime in the current period that are observed by the agent but not observed by the researcher. Similarly, the continuous-choice-specific variable η represents the unobserved costs of extracting water in the current period that are observed by the agent but not the researcher.

In this example, the researcher is interested in learning about the administrator's payoffs as a function of state variables and the discrete and continuous choice variables. For example, Timmins (2002) cites evidence that administrators price water significantly below marginal cost, which would result in an inefficient allocation of potable water, a very scarce resource in arid regions of the world.

The timing assumptions of our model reflect that the administrator faces uncertainty about the costs of extracting water (η) when the preliminary pricing regime decision is made. Furthermore, these costs may be correlated with unobserved factors that influence the regime choice (ε).

Example 2. Consider a dynamic model in which each period a firm first chooses whether or not to operate or idle a plant in the current period, $D \in \mathcal{D} = \{0, 1\}$, after observing choice-specific shocks ε_0 and ε_1. Then, conditional on operating the plant and observing the realization of a cost shock η, the firm chooses a quantity of output to produce, $C \in \mathcal{C} = [0, \infty)$. At the time the operation decision D is made, the firm may still have some remaining uncertainty about what it's actual realized costs would be should it choose to operate. The unknown component of costs is represented by the cost shock η. However, the firm knows the distribution of η and chooses D to maximize the expectation of its present discounted profits before η is realized.

Suppose that there is learning-by-doing in the sense that the marginal cost is decreasing in past cumulative production. Suppose further that the state vector is $S = (Q, X)$, where Q is the past cumulative production at the plant, which evolves as $Q_{t+1} = Q_t + C_t$, and X is the vector of other state variables.

The main focus of the paper is on nonparametric identification of the payoff function. However, for the sake of concreteness, consider a model with known inverse demand $p(c, x)$ and linear marginal cost $\alpha - \gamma Q - \eta$, where α is the baseline marginal cost, γ is the amount by which an additional unit of past cumulative production decreases the marginal cost, and η captures unobservable factors that influence costs. The resulting period payoff for $D = 1$ is

$$(4) \qquad \pi(c, s, \eta) = c[p(c, x) - (\alpha - \gamma Q - \eta)].$$

Suppose that the firm must pay a fixed cost ϕ to maintain the plant while idle, so the payoff for $D = 0$ is $-\phi$. No continuous choice is made following the decision to idle. Then, the full period payoff for the firm is

$$U(d, c, s, \varepsilon, \eta) = \begin{cases} -\phi + \varepsilon_0 & \text{if } d = 0, \\ \pi(c, s, \eta) + \varepsilon_1 & \text{if } d = 1. \end{cases}$$

In this example, additive separability (Assumption 1) is satisfied by construction. Conditional independence (Assumption 2) limits the persistence of both the discrete- and continuous-choice-specific unobservables: current values of these unobservables may influence future values, but only through the choices themselves the effects of those choices on the observable state variables.

Before proceeding to our identification results, we discuss our results in light of the well-known under-identification results of Rust (1994) and Magnac and Thesmar (2002). Namely, under certain assumptions, models of dynamic discrete decision processes are nonparametrically unidentified unless the following are all known: the discount factor, the distribution of unobservables, and current and future preferences relative to one alternative. In light of Magnac and Thesmar (2002)'s result that a particular exclusion restriction can be used to identify the discount factor, we do not consider identification of the discount factor (β) in this paper. Although we do provide conditions for nonparametric identification of the payoff function and the distribution of unobservables, our results do not contradict the aforementioned under-identification results due to differences in our model and assumptions. We show that with continuous variation in certain state variables, it is possible to identify the distribution of unobservables. In contrast, Rust (1994) and Magnac and Thesmar (2002) considered a model with only discrete state variables. We also establish identification of the payoff function subject to a location restriction, which is weaker than than assuming the payoff for one discrete choice is identically zero, as previous authors have done. On the other hand, Rust (1994) did not consider such a condition. We will discuss each of our assumptions in more detail below after stating them formally.

3. Identification of the Payoff Function

In this section, we focus on nonparametric identification of the continuous choice policy function σ and the payoff function u given that the distribution of unobservables is known or identified. We then consider nonparametric identification of the distribution of unobservables in the following section.

3.1. Identification of σ

For given values of d, s, and η, let $\sigma(d,s,\eta)$ be the value of the continuous choice that solves the maximization problem inside the expectation in (2). We maintain the following monotonicity assumption on σ.

Assumption 5 (Monotone Choice)**.** For all d and s, $\sigma(d,s,\eta)$ is strictly increasing in η.

Under Assumption 5, $\sigma(d,s,\cdot)$, is a one-to-one[4] mapping from the space of private shocks $\mathcal{H}$ to the space of continuous choices $\mathcal{C}$, for all d and s. Similar monotonicity conditions have been widely used both in empirical work and in identification analysis in related models, including, but not limited to Matzkin (2003), Chesher (2003), Benkard and Bajari (2005), Bajari et al. (2007), Hong and Shum (2010), and Srisuma (2013). Lemmas 2 and 3 stated in Appendix B provide two more primitive, sufficient conditions on u for this monotonicity assumption.

Example 1, Continued. In the context of Example 1, the monotone choice assumption requires that the optimal price of water, in both pricing regimes d and in all states s, is monotonically increasing in the unobserved costs of extraction. Indeed, in the actual parametric specification Timmins (2002) used for his application the policy function is monotonically increasing in the unobservable.

Example 2, Continued. In the dynamic production model of Example 2, the monotone choice assumption requires that the output quantity is strictly increasing in the productivity shock. As shown in Lemma 3 (Appendix B), if $\frac{\partial^2 u(d,c,s,\eta)}{\partial c \partial \eta} > 0$, then Assumption 5 holds. Since the example model is parametric, this can be verified directly: $\frac{\partial^2 \pi}{\partial c \partial \eta} = 1 > 0$.

The following theorem establishes identification of σ under Assumption 5 given $F_{\eta|D,S}$. We consider identification of $F_{\eta|D,S}$ in the following section.

Theorem 1. *Let Assumptions 1–5 hold. If $F_{\eta|D,S}$ is known or identified, then σ is identified.*

Proof. Let $d \in \mathcal{D}$ and $s \in \mathcal{S}$ be given. Under Assumption 5, the inverse policy function is related to the conditional cdf of the observed continuous choice as follows:

$$\begin{aligned} F_{C|D=d,S=s}(c) &= \Pr(C \leq c \mid D=d, S=s) = \Pr\left(\sigma(d,s,\eta) \leq c \mid D=d, S=s\right) \\ &= \Pr\left(\eta \leq \sigma^{-1}(d,s,c) \mid D=d, S=s\right) = F_{\eta|D=d,S=s}\left(\sigma^{-1}(d,s,c)\right). \end{aligned}$$

[4]Because the payoff function depends on η in a nonseparable manner, the sign of η is not identified. Strict monotonicity of σ in η is the key requirement. The assumption that σ is strictly *increasing* in η (as opposed to strictly *decreasing*) is a normalization: if σ is strictly decreasing in η, then it is strictly increasing in $-\eta$.

The first equality follows by definition of the conditional cdf, the second by definition of C, the third by monotonicity, and the fourth again by definition of the conditional cdf. Then $\sigma^{-1}(d,s,c) = F^{-1}_{\eta|D=d,S=s}\left(F_{C|D=d,S=s}(c)\right)$ is identified. Finally, for all d and s, since $\sigma^{-1}(d,s,\cdot)$ is strictly increasing, then $\sigma(d,s,\cdot)$ is identified. Since d and s were arbitrary, σ is identified on $\mathcal{D} \times \mathcal{S} \times \mathcal{H}$. ■

Remark. Note that $F_{\eta|D,S}$ can be obtained if the joint conditional distribution $F_{\varepsilon,\eta|S}$ and the function Δv are known. The former may be specified parametrically in some cases while the latter can often be obtained from the (observable) conditional choice probabilities (Hotz and Miller, 1993; Rust, 1994). Let $\mathcal{E}_{d,s}$ be the region of $\mathcal{E}$ corresponding to the optimal choice of $D = d$ given that the state is $S = s$:[5]

$$\mathcal{E}_{d,s} = \{\varepsilon \in \mathcal{E} : v(d,s) + \varepsilon_d > v(k,s) + \varepsilon_k \; \forall k \neq d\}.$$

If the function Δv is identified, then so are the regions $\mathcal{E}_{d,s}$ for all d and s. Then, the cdf in question is identified since

$$F_{\eta|D=d,S=s}(h) = \frac{\Pr(\eta \leq h, D = d \mid S = s)}{\Pr(D = d \mid S = s)} = \frac{\Pr(\eta \leq h, \varepsilon \in \mathcal{E}_{d,s} \mid S = s)}{\Pr(D = d \mid S = s)}$$

and since $F_{D,S}$ and $F_{\varepsilon,\eta|S}$ are both known.

3.2. Identification of u

Let $\mathcal{U} = \mathcal{D} \times \mathcal{C} \times \mathcal{S} \times \mathcal{H}$ be the domain of u. The following theorem establishes conditions under which the payoff function u is identified on a particular subset of its domain, $\widetilde{\mathcal{U}} \subseteq \mathcal{U}$, which corresponds to potentially observable optimal behavior by the agent. The two corollaries that follow provide additional conditions—an additive separability condition and an exclusion restriction—either of which suffices for the utility function to be nonparametrically identified on the entire domain $\mathcal{U}$.

Assumption 6 (Discount Factor)**.** The discount factor $\beta \in [0,1)$ is known.

Assumption 7 (Regularity Conditions)**.** u is continuous, bounded, and differentiable. For all d, s, and η, u is such that $\partial u(d,s,c,\eta)/\partial c > 0$, $\lim_{c\to\inf \mathcal{C}} \partial u(d,s,c,\eta)/\partial c = \infty$, and $\lim_{c\to\sup \mathcal{C}} \partial u(d,s,c,\eta)/\partial c = 0$. For all continuous, bounded functions $g : \mathcal{S} \to \mathbb{R}$, $\mathrm{E}[g(S') \mid D = d, C = c, S = s] < \infty$ for all d, c, and s.

Assumption 8 (Period Utility)**.** For some $\overline{d} \in \mathcal{D}$, $u(\overline{d}, c, s, \eta)$ is known for all c, s, and η. For some $\overline{c}$, $u(d, \overline{c}, s, \overline{\eta})$ is known for all d, s, and $\overline{\eta} = \sigma^{-1}(d,s,\overline{c})$.

[5] We ignore ties here, which happen with probability zero under Assumption 4.

Assumption 6 requires that the discount factor is known by the researcher or that it has been otherwise identified since, as discussed previously, we do not consider identification of β. Assumption 7 contains several regularity conditions which will be used to guarantee an interior solution for the continuous choice.[6] This assumption is invoked because we use the first-order condition from the agent's optimization problem and require the existence of a fixed point to a certain functional operator used in the proofs.

Assumption 8 is needed because, unlike in the static discrete choice case, not even differences in the period utility function are identified. We maintain the focus on nonparametric identification of u with the caveat that this assumption requires some prior information about u. However, this assumption is weaker than either assuming the parametric family of functions in which u lies is known or that the payoff for one alternative is identically zero, both of which are assumptions that have been invoked previously in the literature. As in Aguirregabiria (2010), it may be possible to directly identify certain counterfactual implications without first identifying the level of u, in which case this assumption may be unnecessary.

Theorem 2. *If Assumptions 1–8 are satisfied and $F_{\eta|D,S}$ and $F_{\varepsilon|S}$ are known or identified, then u is identified on a subset of the domain $\widetilde{\mathcal{U}} \subseteq \mathcal{U}$, where*

$$\widetilde{\mathcal{U}} = \left\{ (d, c, s, \eta) : d \in \mathcal{D}, c \in \mathcal{C}, s \in \mathcal{S}, \eta = \sigma^{-1}(d, s, c) \right\}.$$

Remark. The conclusion of this theorem is equivalent to showing that the function $\tilde{u}$ is nonparametrically identified where $\tilde{u}(d, c, s) \equiv u(d, c, s, \sigma^{-1}(d, s, c))$ for all d, c, and s. Intuitively, without additional assumptions the domain of identification is limited because we never observe suboptimal behavior by the agent and so the observables do not contain information about u over regions where $c \neq \sigma(d, s, \eta)$.

We reserve the proof of this theorem for the appendix, but provide a brief outline here. The structure of discrete and continuous choice models can be complex, but our main result exploits the particularly tractable structure that follows from Assumptions 1–3. This allows us to build on the insights of Hotz and Miller (1993) for discrete choice models to establish a one-to-one mapping between the (potentially observable) conditional choice probabilities and differences in the discrete-choice-specific value function. We then identify the level of v for the normalizing choice $\overline{d}$ by showing that it is the unique fixed point of a particular functional mapping that depends only on identified quantities. Finally, the payoff function itself is identified by using the first-order condition for optimality of the agent's continuous choice and the monotonicity assumption, which relates the observable continuous choices to the unobservable private shocks in a tractable manner.

[6] Although this assumption limits our analysis to interior solutions, this is mitigated by the existence of the preliminary discrete choice since one of the choices can serve to indicate a corner solution.

We now state and prove two corollaries for extending identification of u from $\widetilde{\mathcal{U}}$ to the entire domain $\mathcal{U}$. The first corollary provides an additive separability condition, which allows for interactions between the observables (d, c, s) and the unobservable η as long as the functional form of the interaction is known. This is a generalization of the type of additive separability assumption used for ε (Assumption 1) and is analogous to assuming the partial derivative of u with respect to η is known. As we illustrate below, this is assumption is satisfied in many applications. For example, it holds when c is quantity and η is a marginal cost shock.

Corollary. *In addition to the assumptions of Theorem 2, suppose that for all $(d, s, c, \eta) \in \mathcal{U}$,*

$$u(d, c, s, \eta) = u_1(d, c, s) + u_2(d, c, s) \cdot \eta, \tag{5}$$

where the function u_2 is known. Then u is identified on $\mathcal{U}$.

Proof. Suppose u is known on the set $\widetilde{\mathcal{U}}$ and that for each $(d, c, s, \eta) \in \mathcal{U}$ we can write $u(d, c, s, \eta) = u_1(d, c, s) + u_2(d, c, s) \cdot \eta$, where u_2 is known. Note that

$$u_1(d, c, s) = u(d, c, s, \eta) - u_2(d, c, s) \cdot \eta \tag{6}$$

for any $\eta \in \mathcal{H}$. Let $(d, s, c) \in \mathcal{U}_1$ and choose $\eta = \sigma^{-1}(d, s, c)$. Then $(d, s, c, \eta) \in \widetilde{\mathcal{U}}$, so $u(d, c, s, \eta)$ is known. Then $u_1(d, s, c)$ is identified from (6) and $u(d, s, c, \eta)$ is in turn identified from (5). ■

Example 2, Continued. Note that the payoff function of Example 2 satisfies the necessary multiplicative separability condition, as seen in (4). Since η is a marginal cost shock, it enters the payoff function multiplicatively with the continuous choice c. Hence, the payoff function has the form required by (6) with $u_2(d, c, s) = c$. The remainder of the payoff function, denoted u_1, would then be nonparametrically identified on its entire domain.

The second corollary provides a different condition under which it is possible to extrapolate the shape of the payoff function to the entire domain. This additional condition is a simple exclusion restriction which does not presume additional knowledge about the functional form of u. Again, as the example below illustrates, this condition is not difficult to satisfy in practice.

Assumption 9. The state vector can be written as $S = (S_0, S_1)$ where S_1 does not affect the period payoff directly, but affects the optimal continuous choice. That is, where $u(d, c, s, \eta) = u(d, c, s', \eta)$ for all $d, c, s, s' \in \mathcal{S}$ such that $s = (s_0, s_1)$ and $s' = (s_0, s'_1)$ for some s_0, s_1, and s'_1. Furthermore, for any d, c, η, and s_0, $\sigma(d, s, \eta) = c$ for some value of s_1.

Corollary. *In addition to the assumptions of Theorem 2, if Assumption 9 holds, then the function u is nonparametrically identified on its entire domain $\mathcal{U}$.*

Proof. Let $(d, c, s, \eta) \in \mathcal{U}$ with $s = (s_0, s_1)$ and recall that σ is identified by Theorem 1 under the maintained assumptions. Since this is an arbitrary element of $\mathcal{U}$, it may be such that $c \neq \sigma(d, s, \eta)$ and hence it may not be an element of $\widetilde{\mathcal{U}}$. We proceed by constructing a corresponding element of $\widetilde{\mathcal{U}}$ for which the utility is identified (by Theorem 2) and known to be equal to that for the arbitrary chosen point in $\mathcal{U}$. Under Assumption 9, there exists an $\tilde{s}_1 \in \mathcal{S}_1$ such that $\sigma(d, c, \tilde{s}, \eta) = c$ for $\tilde{s} = (s_0, \tilde{s}_1)$. Then $u(d, c, s, \eta) = u(d, c, \tilde{s}, \eta)$, also by Assumption 9. By construction, $(d, c, \tilde{s}, \eta) \in \widetilde{\mathcal{U}}$, and $u(d, c, \tilde{s}, \eta)$ is identified by Theorem 2. It follows that $u(d, c, s, \eta)$ is identified. ■

Example 1, Continued. In the context of Example 1, the exclusion restriction is satisfied by variables that act as demand shifters. For example, Timmins (2002) estimates a demand function for water that controls for income and the number of service connections in the municipality. So, S_0 would include variables that do affect payoffs, such as the lift-height (the distance water from the aquifer must be lifted to the surface), and S_1 would include variables that are excluded from the payoff function, but affect the pricing decision because they determine the demand for water, such as income and the number of connections.

4. Identification of the Distribution of Unobservables

The previous section considered nonparametric identification of u given that $F_{\varepsilon,\eta|S}$ was known or identified. In this section, we take a brief look nonparametric identification of $F_{\varepsilon,\eta|S}$ itself. Our work here follows a rich literature on identification and estimation of static discrete choice models, which spans the range of parametric, semiparametric, and nonparametric models. McFadden (1974) initially considered a parametric model where the distribution of ε was known. A literature on semiparametric estimation of binary choice models and related models ensued, including the estimators of Manski (1975, 1985), Cosslett (1983), Stoker (1986), Han (1987), Horowitz (1992, 1996), Ichimura (1993), and Klein and Spady (1993), among many others. Manski (1988) compared the identifying power of many types of assumptions in semiparametric binary choice models more generally. Dubin and McFadden (1984) estimated a parametric, static model with both discrete and continuous outcomes and Newey (2007) studied nonparametric identification of the model in the binary choice, however, the timing and informational assumptions are different in our model.

Below, we will consider the identifying power of conditions that are most closely related to those used by Matzkin in work on semiparametric identification of discrete choice models (Matzkin, 1991), nonparametric identification of the distribution of unobservables for binary and multinomial choice models (Matzkin, 1992, 1993), and nonparametric identification in nonseparable models with continuous outcomes (Matzkin, 2003). Her findings will prove useful in the dynamic model we consider, but as we illustrate below they are not directly applicable due to the recursive structure of the dynamic payoff functions the model we consider.

There have also been promising recent developments on nonparametric identification of the distribution of unobservables in dynamic discrete choice models. Norets and Tang (2012) develop a semiparametric estimator for dynamic binary choice models that does not require the distribution of unobservables to be known, but they do not consider nonparametric identification of this distribution. In another line of work, Heckman and Navarro (2007) and Aguirregabiria (2010) established nonparametric identification of the distribution of discrete-choice-specific unobservables when observations on a continuous outcome variable are available (e.g., when revenues are observed in the dynamic production example) and certain restrictions on the unobservables are satisfied. Although we consider models for which the only observable continuous outcome is the continuous choice variable, which is a decision variable, if another continuous outcome variable is available, then identification may be possible under different assumptions than those we consider below. The location restriction of Assumption 8 plays a similar role in the proofs for the present model.

4.1. Identification of $F_{\eta|D,S}$

We first consider identification of $F_{\eta|D,S}$ and then turn to identification of $F_{\varepsilon|S}$. Our results on identification of $F_{\eta|D,S}$ rely on the following conditional independence assumption.

Assumption 10. $S = (S_0, S_1)$ and $\eta \perp\!\!\!\perp S_1 \mid D, S_0$. That is, for all h, $F_{\eta|D,S}(h) = F_{\eta|D,S_0}(h)$ almost surely.

Under Assumption 10, for all d, s_0, and s_1, we have

$$
\begin{aligned}
(7)\qquad F_{\eta|D=d,S_0=s_0}(h) &= \Pr(\eta \leq h \mid D = d, S_0 = s_0) = \Pr(\eta \leq h \mid D = d, S = s) \\
&= \Pr(\sigma(d,s,\eta) \leq \sigma(d,s,h) \mid D = d, S = s) = F_{C|D=d,S=s}(\sigma(d,s,h))
\end{aligned}
$$

for all $h \in \mathcal{H}$. The first equality follows by definition of the conditional cdf, the second by conditional independence (Assumption 10), the third by monotonicity of σ (Assumption 5), and the fourth by definition of C.

Since $F_{C|D,S}$ is identified, the relationship in (7) suggests that if there are a priori restrictions on σ suggested by economic theory, then we may be able to identify $F_{\eta|D,S}$. One such restriction is homogeneity of degree zero.

Assumption 11. Suppose that each choice $d \in \mathcal{D}$ there exist values $\bar{h} \in \mathcal{H}$, $\alpha \in \mathbb{R}$, and $\bar{s}_1 \in \mathcal{S}_1$ such that for all $s_0 \in \mathcal{S}_0$ and $\lambda \in \mathbb{R}$ we have $\sigma(d, s_0, \bar{s}_1, \bar{h}) = \sigma(d, s_0, \lambda\bar{s}_1, \lambda\bar{h}) = \alpha$.

Theorem 3 requires homogeneity of degree zero along a given ray in some subset of the arguments. If σ is an output supply function, and η is a marginal cost shock, as is the case in Example 2, then economic theory would suggest that σ should be homogeneous of degree zero in some arguments, such as input and output prices and η. Identification of $F_{\eta|D,S}$ is also possible under different conditions, when applicable, such as those considered by Matzkin (2003), including homogeneity of degree one.

Theorem 3. *If Assumptions 1–5, 10, and 11 hold, then $F_{\eta|D,S}$ is identified.*

Proof. Let d, s_0, and λ be given and let $\bar{s}_1$, h, and α satisfy Assumption 11. Let $s_1 = \lambda \bar{s}_1$ and $h = \lambda \bar{h}$. Then

$$F_{\eta|D=d,S_0=s_0}(\lambda\bar{h}) = F_{C|D=d,S_0=s_0,S_1=\lambda\bar{s}_1}(\sigma(d, s_0, \lambda\bar{s}_1, \lambda\bar{h})) = F_{C|D=d,S_0=s_0,S_1=\lambda\bar{s}_1}(\alpha).$$

The first equality follows from (7) and the second holds by Assumption 11, under which $\sigma(d, s_0, \lambda\bar{s}_1, \lambda\bar{h}) = \alpha$. It follows that for any $h \in \mathcal{H}$,

$$F_{\eta|D=d,S_0=s_0}(h) = F_{C|D=d,S_0=s_0,S_1=\tilde{s}_1}(\alpha),$$

where $\tilde{s}_1 = (h/\bar{h})\bar{s}_1$ and where we have chosen $\lambda = h/\bar{h}$. Therefore, since $F_{C|D,S}$ is identified, so is $F_{\eta|D=d,S_0=s_0}(h)$. Since d and s_0 were arbitrary and since η is conditionally independent of S_1, it follows that $F_{\eta|D,S}$ is identified. ■

4.2. Identification of $F_{\varepsilon|S}$

Now, we turn to identification of $F_{\varepsilon|S}$. Because only differences in payoffs are relevant, at best we will be able to identify the distribution of $\Delta\varepsilon$ given S. Previous authors have considered semiparametric or nonparametric identification in related finite horizon models (Taber, 2000; Heckman and Navarro, 2007; Aguirregabiria, 2010), but as illustrated by the non-identification result of Rust (1994), establishing nonparametric identification in the infinite-horizon case presents additional challenges. Abstracting from the continuous choice, the results below also constitute a new identification result for the distribution of discrete-choice-specific unobservables in infinite-horizon dynamic discrete choice models.

We note that our assumptions are different from those of Rust (1994) in two key ways. First, we will assume that ε is independent of some components of S while Rust (1994) allowed the distribution of ε to depend on the entire state vector S. Additionally, in the model of Rust (1994) and Magnac and Thesmar (2002), the observable state variables have discrete support. On the other hand, to identify the entire distribution, we require a variable with continuous support for each choice that can be excluded from the payoffs of the remaining choices.

To motivate our identification result, first note that since both D and S are observable, the conditional choice probabilities are identified. Note further that these choice probabilities are related to $F_{\Delta\varepsilon|S}$ as follows:

$$\text{(8)} \quad \Pr(D = 0 \mid S) = F_{\Delta\varepsilon|S}(-\Delta v(1, S), \ldots, -\Delta v(K, S)).$$

This relationship provides the basis for our identification result.

The identification conditions we consider are choice-specific exclusion restrictions. Suppose that there exist $K + 1$ choice-specific variables $Z_0, Z_1, \ldots, Z_K$ such that for each choice d, one of these variables affects the payoff for choice d, but not the other choices. There may be other choice-specific variables, but they need not satisfy the exclusion restriction, so they can be included in a common state vector X. More specifically, suppose that the mean payoff for choice d in state $s = (x, z_0, z_1, \ldots, z_K)$ can be written as $u(d, c, s, \eta) = u(d, c, x, z_d, \eta)$ for some choice-specific variable z_d and common variables x. Furthermore, we assume that $\Delta\varepsilon$ and η are independent of $Z_0, Z_1, \ldots, Z_K$ but may depend on X.

Our identification strategy is to use these exclusion restrictions to vary $Z_0, Z_1, \ldots, Z_K$ in such a way as to vary the K arguments of the cdf $F_{\Delta\varepsilon|S}$ in a manner that will allow us to recover the value of the cdf by relating it to the observable conditional choice probabilities using (8). Recall that we can separately identify σ using, for example, Theorem 1, so we may take it as given if the relevant assumptions are satisfied. Focusing on the arguments of the cdf, note that for each d and s, we can decompose $\Delta v(d, s)$ as follows:

$$\begin{aligned}
\Delta v(d, s) &= \mathrm{E}\left[\max_{c \in \mathcal{C}} \left\{ u(d, c, x, z_d, \eta) + \beta\, \mathrm{E}\left[\bar{V}(s') \mid D = d, C = c, S = s\right] \right\} \,\middle|\, D = d, S = s\right] \\
&\quad - \mathrm{E}\left[\max_{c \in \mathcal{C}} \left\{ u(0, c, x, z_0, \eta) + \beta\, \mathrm{E}\left[\bar{V}(s') \mid D = 0, C = c, S = s\right] \right\} \,\middle|\, D = 0, S = s\right] \\
&= \mathrm{E}[u(d, \sigma(d, s, \eta), x, z_d, \eta) \mid D = d, S = s] \\
&\quad - \mathrm{E}[u(0, \sigma(0, s, \eta), x, z_0, \eta) \mid D = 0, S = s] \\
&\quad + \beta\, \mathrm{E}\left[\mathrm{E}\left[\bar{V}(s') \mid D = d, C = \sigma(d, s, \eta), S = s\right] \mid D = d, S = s\right] \\
&\quad - \beta\, \mathrm{E}\left[\mathrm{E}\left[\bar{V}(s') \mid D = 0, C = \sigma(0, s, \eta), S = s\right] \mid D = 0, S = s\right]
\end{aligned}$$

Choice-specific exclusion restrictions can be used for identification in the static case (Matzkin, 1993), however, from above one can see that the situation in the dynamic case requires that we address the future value terms. Taken alone, such restrictions do not have the same identifying power in the dynamic case because the agent is forward-looking. In the static case, with $\beta = 0$, one could change z_d and change only the period payoff function for choice d. However, in the dynamic case changing z_d in the current period may influence future realizations of z_d or even z_k for alternatives $k \neq d$. Hence, even though the period payoff for choice d only depends on x and z_d, for each choice d the differenced value function $\Delta v(d,s)$ still varies with all components of s. This confounds attempts to vary the arguments of the cdf $F_{\Delta\varepsilon|S}$ in (8) in a tractable manner, a common identification strategy in static models.

For simplicity, we will focus on the case of a binary choice model with $K = 1$ and $\mathcal{D} = \{0, 1\}$ and, now dropping the choice argument, we define $\Delta v(s) \equiv v(1,s) - v(0,s)$ and $\Delta\varepsilon \equiv \varepsilon_0 - \varepsilon_1$. In this case, for any state s the choice probability for $D = 1$ is

$$P(s) \equiv \Pr(D = 1 \mid X = x, Z_0 = z_0, Z_1 = z_1) = F_{\Delta\varepsilon|X=x}(\Delta v(x, z_0, z_1)).$$

We show that it is possible to identify $F_{\Delta\varepsilon|X}$ in our model using the exclusion restrictions discussed above provided that the persistence of the excluded variables is limited.

Assumption 12. $S = (X, Z_0, Z_1)$ where $\Delta\varepsilon, \eta \perp\!\!\!\perp Z_0, Z_1 \mid X$ and $\mathrm{Med}(\Delta\varepsilon \mid X) = 0$ almost surely.

Assumption 13. For all $c \in \mathcal{C}$, $x \in \mathcal{X}$, $z_0 \in \mathcal{Z}_0$, $z_1 \in \mathcal{Z}_1$, and $h \in \mathcal{H}$, the payoff function is such that $u(0, c, x, z_0, z_1, h) = u(0, c, x, z_0, \tilde{z}_1, h)$ for all $\tilde{z}_1 \in \mathcal{Z}_1$ and $u(1, c, x, z_0, z_1, h) = u(1, c, x, \tilde{z}_0, z_1, h)$ for all $\tilde{z}_0 \in \mathcal{Z}_0$.

Assumption 14. $Z_0', Z_1' \perp\!\!\!\perp Z_0, Z_1 \mid X, D, C$.

Assumption 12 requires that the distribution of $\Delta\varepsilon$ and η conditional on X be identical for all Z_0 and Z_1 and that the conditional median of $\Delta\varepsilon$ given X is zero. Under Assumption 12, for all $s = (x, z_0, z_1)$ the choice probability for $D = 1$ can be written as

$$P(x, z_0, z_1) = F_{\Delta\varepsilon|X=x, Z_0=z_0, Z_1=z_1}(\Delta v(x, z_0, z_1)) = F_{\Delta\varepsilon|X=x}(\Delta v(x, z_0, z_1)).$$

Assumption 13 formalizes the previously mentioned payoff exclusion restrictions. Assumption 14 requires two choice-specific variables that are serially independent conditional on the common variables X and the choice variables D and C.

Theorem 4. *Let Assumptions 1–8 and 12–14 hold and let $K = 1$. Suppose $F_{\eta|D,S}$ and σ are known or identified. Then, for each $s = (x, z_0, z_1) \in \mathcal{S} = \mathcal{X} \times \mathcal{Z}_0 \times \mathcal{Z}_1$, $F_{\Delta\varepsilon|S=s}(e)$ is identified for all $e \in \Delta v(x, \mathcal{Z}_0, \mathcal{Z}_1)$.*

Finally, combining the results of Theorems 1–4 yields full nonparametric identification of the model. Table 1 summarizes the assumptions and primitives required for each step. Using Theorem 3, we can first identify $F_{\eta|D,S}$, which can in turn be used to identify σ using Theorem 1. Then, given $F_{\eta|D,S}$ and σ, we can identify $F_{\varepsilon|S}$ using Theorem 4 and u using Theorem 2. Naturally, one also has the option of substituting parametric distributional assumptions for either $F_{\eta|D,S}$ or $F_{\varepsilon|S}$ and the associated assumptions to achieve semi-nonparametric identification.

Theorem	Assumptions	Given Primitives	Identified Primitive
Theorem 3	1–5, 10–11	–	$F_{\eta\|D,S}$
Theorem 1	1–5	$F_{\eta\|D,S}$	σ
Theorem 4	1–5, 6–8, 12–14	$F_{\eta\|D,S}$, σ	$F_{\varepsilon\|S}$
Theorem 2	1–5, 6–8	$F_{\eta\|D,S}$, σ, $F_{\varepsilon\|S}$	u

TABLE 1. Summary of Assumptions and Theorems

5. Conclusion

We have established conditions for nonparametric identification of the payoff functions and distributions of unobservables in dynamic decision processes in which agents make both discrete and continuous choices. Such models are widely used in applied work in economics and one goal of our analysis is to provide a point of reference for practitioners in specifying and estimating parametric instances of the model considered. To this end, we have examined the identifying power of several assumptions and attempted to find conditions which are based on either commonly available sources of variation in the observables or structural restrictions suggested by economic theory or currently used in applied work.

Identification analysis involves trade-offs among various assumptions, and our discussion of nonparametric identification of the distribution of unobservables highlights the importance of continuous variation in the excluded variables and addresses difficulties faced when extending existing identification conditions from static models to infinite-horizon dynamic models. Our results on identification of the distribution of discrete-choice-specific unobservables also constitute a new result for dynamic, pure discrete choice models.

Our work also suggests a number of interesting extensions for future research. This includes considering identification of various counterfactual implications without normalizing the payoff function and studying identification of $F_{\varepsilon|S}$ in cases with allow for more general forms of serial correlation in the excluded variables.

A. Proofs

Lemma 1. *Suppose the assumptions of Theorem 2 hold. Let $C(\mathcal{S})$ denote the Banach space of all continuous, bounded functions $w : \mathcal{S} \to \mathbb{R}$ under the supremum norm, $\|w\| = \sup_{s \in \mathcal{S}} |w(s)|$.*

Let $\Gamma : C(\mathcal{S}) \to C(\mathcal{S})$ *be the functional operator defined as*

$$\Gamma w(s) \equiv \mathrm{E}\left[\max_{c\in\mathcal{C}}\left\{u(\bar{d},c,s,\eta) + \beta\,\mathrm{E}\left[H(S') + w(S') \mid \bar{d},c,s\right]\right\}\,\middle|\,\bar{d},s\right]$$

Then, Γ *is a contraction with modulus* β*.*

Proof of Lemma 1. We will show that for any two functions $w,\tilde{w} \in C(\mathcal{S})$, $\|\Gamma w - \Gamma\tilde{w}\| \le \kappa\,\|w - \tilde{w}\|$ for some $\kappa \in (0,1)$. For notational brevity, define $\psi(c,s,\eta) = u(\bar{d},c,s,\eta) + \beta\,\mathrm{E}[H(S') \mid \bar{d},c,s]$. We have

$$\begin{aligned}
\|\Gamma w - \Gamma\tilde{w}\| &= \sup_{s\in\mathcal{S}}\left|\mathrm{E}\left[\max_{c\in\mathcal{C}}\left\{\psi(c,s,\eta) + \beta\,\mathrm{E}[w(S') \mid \bar{d},c,s]\right\}\,\middle|\,\bar{d},s\right]\right. \\
&\qquad\left. - \mathrm{E}\left[\max_{c\in\mathcal{C}}\left\{\psi(c,s,\eta) + \beta\,\mathrm{E}[\tilde{w}(S') \mid \bar{d},c,s]\right\}\,\middle|\,\bar{d},s\right]\right| \\
&\le \sup_{s\in\mathcal{S}}\mathrm{E}\left[\left|\max_{c\in\mathcal{C}}\left\{\psi(c,s,\eta) + \beta\,\mathrm{E}[w(S') \mid \bar{d},c,s]\right\}\right.\right. \\
&\qquad\left.\left. - \max_{c\in\mathcal{C}}\left\{\psi(c,s,\eta) + \beta\,\mathrm{E}[\tilde{w}(S') \mid \bar{d},c,s]\right\}\right|\,\middle|\,\bar{d},s\right] \\
&\le \beta\sup_{s\in\mathcal{S}}\left|\max_{c\in\mathcal{C}}\mathrm{E}[w(S') - \tilde{w}(S') \mid \bar{d},c,s]\right| \\
&\le \beta\sup_{s'\in\mathcal{S}}\left|w(s') - \tilde{w}(s')\right| \\
&= \beta\,\|w - \tilde{w}\|\,.
\end{aligned}$$

The first equality follows from the definitions of the norm and the functional operator. The second line follows from the linearity of the conditional expectation operator and a property of absolute values of integrals: $\left|\int f\right| \le \int |f|$. The third line follows from the a property of the max operator, $|\max f - \max g| \le \max|f - g|$, from linearity of the (inner) conditional expectation, and by noting that the outer expectation was over η and all terms involving η cancel out. The fourth line follows from Assumption 7. The final equality holds by definition of the norm. ■

Proof of Theorem 2. Without loss of generality, suppose that Assumption 8 holds for $\bar{d} = 0$.

Step 1: Identification of Δv First, note that we can write the discrete choice probabilities in terms of the discrete-choice-specific value function as

$$\Pr(D = k \mid S = s) = \mathrm{E}\left[1\{k = \arg\max_{d\in\mathcal{D}}[v(d,s) + \varepsilon_d]\}\,\middle|\,S = s\right].$$

Under Assumption 4, there exists a one-to-one mapping $\Psi(\cdot;s)$ from the conditional choice probabilities to differences in the discrete-choice-specific value function in state s:

$$(\Delta v(1,s),\ldots,\Delta v(K,s)) = \Psi\left(\Pr(D = 1 \mid S = s),\ldots,\Pr(D = K \mid S = s);s\right),$$

which follows from Lemma 3.1 of Rust (1994), a result of Hotz and Miller (1993). Also see Corollary 1 of Norets and Takahashi (2013) for a stronger result, that $\Psi(\cdot; s)$ is a surjection. This mapping identifies the differences $\Delta v(k,s)$ for each $k = 1, \dots, K$ and $s \in \mathcal{S}$.

Step 2: Identification of v Define the function $\tilde{H} : \mathbb{R}^{K+1} \to \mathbb{R}$ as

$$\tilde{H}(r_0, r_1, \dots, r_K; s) \equiv \mathrm{E}\left[\max_{d \in \mathcal{D}} [r_d + \varepsilon_d] \;\middle|\; S = s \right]. \tag{9}$$

Then $\tilde{H}(v(0,s), \dots, v(K,s); s)$ gives the ex-ante expected utility from making the optimal discrete choice in state s. Under Assumption 4, this function exists and has the following additivity property (Rust, 1994, Theorem 3.1): $\tilde{H}\left(r_0 + \alpha, \dots, r_K + \alpha; s\right) = \tilde{H}\left(r_0, \dots, r_K; s\right) + \alpha$. In particular,

$$\tilde{H}\left(v(0,s), \dots, v(K,s); s\right) = \tilde{H}\left(0, \Delta v(1,s), \dots, \Delta v(K,s); s\right) + v(0,s) \equiv H(s) + v(0,s),$$

where the function

$$H(s) \equiv \tilde{H}\left(0, \Delta v(1,s), \dots, \Delta v(K,s); s\right) \tag{10}$$

is identified since Δv is identified and $F_{\varepsilon|S}$ is known. The dependence of $H(s)$ on $\Delta v(\cdot, s)$ is implicit.

We now show that $v(0,s)$ is identified as the unique fixed point of a functional equation, which we show to be a contraction. Hence, the proof strategy for this step is in the spirit of Bajari et al. (2007), but the functional operator in this case is quite different than in their case. First, by definition of $\bar{V}(s)$ and H, $\mathrm{E}[\bar{V}(s') \mid d, c, s] = \mathrm{E}[H(s') + v(0, s') \mid d, c, s] \equiv \phi(d, c, s)$. Using this identity in (3) and evaluating v at $d = 0$ yields a functional equation for $v(0, \cdot)$:

$$v(0,s) = \mathrm{E}\left[\max_{c \in \mathcal{C}} \left\{ u(0,c,s,\eta) + \beta\, \mathrm{E}\left[H(s') + v(0,s') \mid D = 0, c, s \right] \right\} \;\middle|\; D = 0, s \right].$$

The payoff above is identified by Assumption 8, so everything in this expression is identified except for $v(0, \cdot)$. Lemma 1 establishes that this functional mapping is a contraction and therefore $v(0, \cdot)$ is identified as the unique fixed point by the contraction mapping theorem (cf. Stokey, Lucas, and Prescott, 1989, Theorem 3.2). Since the functions Δv and $v(0, \cdot)$ are both identified, and since we have the identity $v(d,s) = \Delta v(d,s) + v(0,s)$, it follows that v itself is identified.

Step 3: Identification of u It remains to identify the payoff function u. In pure discrete choice models, at this point we can recover u directly from the definition of the choice-specific value function. However, matters are complicated by the addition of the continuous choice due to the additional private shock η and the $\max_{c \in \mathcal{C}}$ operator. The monotone choice assumption allows us to overcome the first problem: the one-to-one relationship between c and η allows us to infer the value of η given values of d, c, and s. We address the second problem by working with the first-order condition (under Assumption 7) and using the payoff normalization (Assumption 8). The agent chooses c in order to maximize $u(d,c,s,\eta) + \beta\phi(d,c,s)$, where ϕ is the conditional expectation defined above, which is now identified. Therefore, under Assumption 7, for given values of d, s, and η, the optimal choice of c satisfies the first-order condition:

$$\frac{\partial}{\partial c} u(d,c,s,\eta) + \beta \frac{\partial}{\partial c}\phi(d,c,s) = 0.$$

Rearranging and using the identity $\eta = \sigma^{-1}(d,s,c)$ for the optimal choice of c given d and s under Assumption 5,

$$\frac{\partial}{\partial c} u(d,c,s,\sigma^{-1}(d,s,c)) = -\beta \frac{\partial}{\partial c}\phi(d,c,s).$$

Then, u is identified on $\widetilde{\mathcal{U}}$ up to Assumption 8 since

$$u(d,c,s,\eta) = u(d,\bar{c},s,\bar{\eta}) - \beta \int_{\bar{c}}^{c} \frac{\partial}{\partial c}\phi(d,z,s)\,dz$$

for all d, c, s, and η such that $\eta = \sigma^{-1}(d,s,c)$. ■

Proof of Theorem 4. Without loss of generality, suppose that $\bar{d} = 0$. First, note that we can express the discrete choice specific value function recursively as follows:

$$\begin{aligned} v(d,s) &= \mathrm{E}\left[\max_{c\in\mathcal{C}}\left\{u(d,c,s,\eta) + \beta\,\mathrm{E}\left[H(s') + v(0,s') \mid d,c,s\right]\right\} \,\middle|\, d,s\right] \\ &= \mathrm{E}\left[u(d,\sigma(d,s,\eta),s,\eta) + \beta\,\mathrm{E}\left[H(s') + v(0,s') \mid d,\sigma(d,s,\eta),s\right] \,\middle|\, d,s\right]. \end{aligned}$$

Furthermore, we can decompose $v(d,s)$ into the following three components,

$$v(d,s) = \bar{u}(d,s) + \beta\phi(d,s) + \beta\psi(d,s),$$

where $\bar{u}(d,s) \equiv \int u(d,\sigma(d,s,h),s,h)\,dF_{\eta|D=d,S=s}(h)$ is the expected utility from choosing d in state s before η is realized, $\phi(d,s)$ is the expected discounted utility obtained from choosing zero in all future periods (i.e., the terms involving $v(0,s)$), and $\psi(d,s)$ is the expected discounted utility from choosing optimally in all future periods less the value of choosing zero (i.e., the terms involving $H(s)$). The terms $\phi(d,s)$ and $\psi(d,s)$ can, in turn, be expressed recursively as

$$\begin{aligned} \phi(d,s) = \iiint & \left[u(0,\sigma(0,s',h'),s',h') + \beta\phi(0,s')\right]\, dF_{\eta'|D=0,S=s'}(h') \\ & \times F_{S'|D=d,C=\sigma(d,s,h),S=s}(s')\,dF_{\eta|D=d,S=s}(h), \\ \psi(d,s) = \iint & \left[H(s') + \beta\psi(0,s')\right]\, dF_{S'|D=d,C=\sigma(d,s,h),S=s}(s')\,dF_{\eta|D=d,S=s}(h). \end{aligned}$$

Importantly, $\phi(d, s)$ is identified for all d and s since β, $F_{\eta|D,S}$, $F_{S'|D,C,S}$, and $u(0, \cdot, \cdot, \cdot)$ are all known or identified under the maintained assumptions.

Now, by continuity of the choice probability function in z_0 and z_1, for all x there exists a path $\zeta_0(x, z_1)$ such that

$$P(x, \zeta_0(x, z_1), z_1) = F_{\Delta\varepsilon|X=x}(\Delta v(x, \zeta_0(x, z_1), z_1)) = 0.5$$

for all z_1. Under the conditional median assumption (Assumption 12), this implies that $\Delta v(x, \zeta_0(x, z_1), z_1) = 0$ for all x and z_1. Applying the decomposition from above and noting that $\overline{u}$ for $d = 1$ does not vary with z_0 due to the exclusion restriction (Assumption 13), this implies that $\overline{u}(1, x, z_1) = -\overline{u}(0, x, \zeta_0(x, z_1)) - \beta\Delta\phi(x, \zeta_0(x, z_1), z_1) - \beta\Delta\psi(x, \zeta_0(x, z_1), z_1)$. Therefore, substituting for $\overline{u}(1, x, z_1)$ for all $s = (x, z_0, z_1)$ we can write

$$\Delta v(x, z_0, z_1) = \overline{\phi}(x, z_0, z_1) + \beta\Delta\psi(x, z_0, z_1) - \beta\Delta\psi(x, \zeta_0(x, z_1), z_1)$$

where

$$\overline{\phi}(x, z_0, z_1) = \beta\Delta\phi(x, z_0, z_1) - \overline{u}(0, x, z_0) - \beta\Delta\phi(x, \zeta_0(x, z_1), z_1) - \overline{u}(0, x, \zeta_0(x, z_1))$$

is an identified function.

Under Assumption 14, for all x, z_0, and z_1, $\Delta\psi(x, z_0, z_1) = \Delta\psi(x, \zeta_0(x, z_1), z_1)$ and so $\Delta v(x, z_0, z_1)$ is identified since $\Delta v(x, z_0, z_1) = \overline{\phi}(x, z_0, z_1)$. Therefore, for any $s = (x, z_1, z_2)$, $F_{\Delta\varepsilon|S=s}$ is identified on $\Delta v(x, \mathcal{Z}_0, \mathcal{Z}_1)$. ■

B. Sufficient Conditions for Assumption 5

The following two lemmas provide sufficient conditions for Assumption 5 to hold. One sufficient condition is strict supermodularity (strictly increasing differences) of u in (c, η). Another sufficient condition is that the cross partial derivative of u with respect to c and η is positive.

Lemma 2. *Let Assumptions 1–4 hold. If for all d and s,*

$$u(d, c', s, \eta') - u(d, c, s, \eta') > u(d, c', s, \eta) - u(d, c, s, \eta)$$

for all $c' > c$ and $\eta' > \eta$, then Assumption 5 holds.

Proof. The policy function for the continuous choice is defined as

$$\sigma(d, s, \eta) = \arg\max_{c \in \mathcal{C}} \left\{ u(d, c, s, \eta) + \beta \, \mathrm{E}[V(S', \varepsilon') \mid d, c, s] \right\}.$$

By Topkis's Theorem (Topkis, 1998), if the objective function (the expression in braces above) is strictly supermodular in (c, η), then the policy function will be strictly increasing in η. Note that in our model, the future value term in this expression does not depend on η, because of the conditional independence assumption. Therefore, it is sufficient for the period payoff function to be strictly supermodular. ■

Lemma 3. *Let Assumptions 1–4 hold. If u is twice continuously differentiable in c and η and $\frac{\partial^2 u(d,c,s,\eta)}{\partial c \partial \eta} > 0$ for all d, c, s, and η, then Assumption 5 holds.*

Proof. We show that the cross partial derivative condition above implies strict supermodularity, which is sufficient for Assumption 5 by the previous Lemma. Since the following arguments hold for all d and s we omit all arguments other than c and η for simplicity. Suppose that $\frac{\partial^2 u(c,\eta)}{\partial c \partial \eta} > 0$. Then $\frac{\partial u(c,\eta)}{\partial \eta} > 0$ is strictly increasing in c. Stated differently, for $c' > c$, $\frac{\partial u(c',\eta)}{\partial \eta} > \frac{\partial u(c,\eta)}{\partial \eta}$ and therefore, $\frac{\partial [u(c',\eta) - u(c,\eta)]}{\partial \eta} > 0$. It follows that $u(c',\eta) - u(c,\eta)$ is an increasing function of η. ■

References

Ackerberg, D., L. Benkard, S. Berry, and A. Pakes (2007). Econometric tools for analyzing market outcomes. In J. J. Heckman and E. E. Leamer (Eds.), *Handbook of Econometrics*, Volume 6A. North Holland.

Aguirregabiria, V. (2010). Another look at the identification of dynamic discrete decision processes: An application to retirement behavior. *Journal of Business and Economic Statistics 28*, 201–218.

Aguirregabiria, V. and P. Mira (2002). Swapping the nested fixed point algorithm: A class of estimators for discrete Markov decision models. *Econometrica 70*, 1519–1543.

Aguirregabiria, V. and P. Mira (2007). Sequential estimation of dynamic discrete games. *Econometrica 75*, 1–53.

Aguirregabiria, V. and P. Mira (2010). Dynamic discrete choice structural models: a survey. *Journal of Econometrics 156*, 38–67.

Bajari, P., C. L. Benkard, and J. Levin (2007). Estimating dynamic models of imperfect competition. *Econometrica 75*, 1331–1370.

Bajari, P., V. Chernozhukov, H. Hong, and D. Nekipelov (2007). Semiparametric estimation of dynamic games of incomplete information. Working paper, University of Minnesota.

Benkard, C. L. and P. Bajari (2005). Hedonic price indexes with unobserved product characteristics, and application to personal computers. *Journal of Business and Economic Statistics 23*, 61–75.

Blevins, J. R. (2010). Nonparametric identification of dynamic games with discrete and continuous choices. Working Paper 10-02, The Ohio State University.

Chesher, A. (2003). Identification in nonseparable models. *Econometrica 71*, 1405–1441.

Cosslett, S. R. (1983). Distribution-free maximum likelihood estimation of the binary choice model. *Econometrica 51*, 765–782.

Dubin, J. A. and D. L. McFadden (1984). An econometric analysis of residential electric appliance holdings and consumption. *Econometrica 52*, 345–362.

Hajivassiliou, V. A. and P. A. Ruud (1994). Classical estimation methods for LDV models using simulation. In R. F. Engle and D. L. McFadden (Eds.), *Handbook of Econometrics*, Volume 4, Chapter 40. Amsterdam: North Holland.

Han, A. K. (1987). Non-parametric analysis of a generalized regression model: The maximum rank correlation estimator. *Journal of Econometrics 35*, 303–316.

Heckman, J. J. and S. Navarro (2007). Dynamic discrete choice and dynamic treatment effects. *Journal of Econometrics 136*, 341–396.

Hong, H. and M. Shum (2010). Pairwise-difference estimation of a dynamic optimization model. *Review of Economic Studies 77*, 273–304.

Horowitz, J. L. (1992). A smoothed maximum score estimator for the binary response model. *Econometrica 60*, 505–531.

Horowitz, J. L. (1996). Semiparametric estimation of a regression model with an unknown transformation of the dependent variable. *Econometrica 64*, 103–137.

Hotz, V. J. and R. A. Miller (1993). Conditional choice probabilities and the estimation of dynamic models. *Review of Economic Studies 60*, 497–529.

Hu, Y. and M. Shum (2012). Nonparametric identification of dynamic models with unobserved state variables. *Journal of Econometrics 171*, 32–44.

Ichimura, H. (1993). Semiparametric least squares (SLS) and weighted SLS estimation of single-index models. *Journal of Econometrics 58*, 71–120.

Jofre-Bonet, M. and M. Pesendorfer (2003). Estimation of a dynamic auction game. *Econometrica 71*, 1443–1489.

Keane, M. P., P. E. Todd, and K. I. Wolpin (2011). The structural estimation of behavioral models: Discrete choice dynamic programming methods and applications. In O. Ashenfelter and D. Card (Eds.), *Handbook of Labor Economics*, Volume 4A, Chapter 4. Amsterdam: North Holland.

Klein, R. W. and R. H. Spady (1993). An efficient semiparametric estimator for binary response models. *Econometrica 61*, 387–421.

Magnac, T. and D. Thesmar (2002). Identifying dynamic discrete decision processes. *Econometrica 70*, 801–816.

Manski, C. F. (1975). Maximum score estimation of the stochastic utility model of choice. *Journal of Econometrics 3*, 205–228.

Manski, C. F. (1985). Semiparametric analysis of discrete response: Asymptotic properties of the maximum score estimator. *Journal of Econometrics 27*, 313–333.

Manski, C. F. (1988). Identification of binary response models. *Journal of the American Statistical Association 83*, 729–738.

Matzkin, R. L. (1991). Semiparametric estimation of monotone and concave utility functions for polychotomous choice models. *Econometrica 59*, 1315–1327.

Matzkin, R. L. (1992). Nonparametric and distribution-free estimation of the binary threshold crossing and the binary choice models. *Econometrica 60*, 239–270.

Matzkin, R. L. (1993). Nonparametric identification and estimation of polychotomous choice models. *Journal of Econometrics 58*, 137–168.

Matzkin, R. L. (2003). Nonparametric estimation of nonadditive random functions. *Econometrica 71*, 1339–1375.

McFadden, D. L. (1974). Conditional logit analysis of qualitative choice analysis. In P. Zarembka (Ed.), *Frontiers in Econometrics*, pp. 105–142. Academic Press.

Newey, W. K. (2007). Nonparametric continuous/discrete choice models. *International Economic Review 48*, 1429–1439.

Norets, A. and S. Takahashi (2013). On the surjectivity of the mapping between utilities and choice probabilities. *Quantitative Economics 4*, 149–155.

Norets, A. and X. Tang (2012). Semiparametric inference in dynamic binary choice models. Economic Theory Center Working Paper 40-2012, Princeton University.

Pakes, A., M. Ostrovsky, and S. Berry (2007). Simple estimators for the parameters of discrete dynamic games (with entry/exit examples). *The RAND Journal of Economics 38*, 373–399.

Pesendorfer, M. and P. Schmidt-Dengler (2007). Asymptotic least squares estimators for dynamic games. *Review of Economic Studies 75*, 901–928.

Rust, J. (1987). Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher. *Econometrica 55*, 999–1013.

Rust, J. (1988). Maximum likelihood estimation of discrete control processes. *SIAM Journal on Control and Optimization 26*, 1006–1024.

Rust, J. (1994). Structural estimation of Markov decision processes. In R. F. Engle and D. L. McFadden (Eds.), *Handbook of Econometrics*, Volume 4, Amsterdam. North Holland.

Shrimpf, P. (2011). Identification and estimation of dynamic games with continuous states and controls. Working paper, University of British Columbia.

Srisuma, S. (2013). Identification in Markov decision models. Working paper, University of Cambridge.

Srisuma, S. and O. Linton (2012). Semiparametric estimation of Markov decision processes with continuous state space. *Journal of Econometrics 166*, 320–341.

Stoker, T. M. (1986). Consistent estimation of scaled coefficients. *Econometrica 54*, 1461–1481.

Stokey, N. L., R. E. Lucas, Jr., and E. C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.

Taber, C. R. (2000). Semiparametric identification and heterogeneity in discrete choice dynamic programming models. *Journal of Econometrics 96*, 201–229.

Timmins, C. (2002). Measuring the dynamic efficiency costs of regulators' preferences: Municipal water utilities in the arid West. *Econometrica 70*, 603–629.

Topkis, D. M. (1998). *Supermodularity and Complementarity*. Princeton University Press.